THE WATERLOO MATHEMATICS REVIEW

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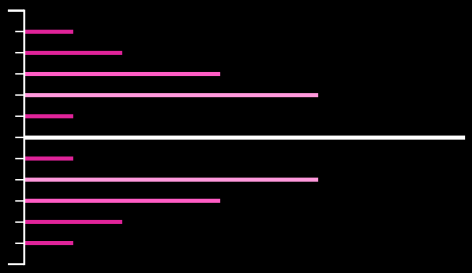

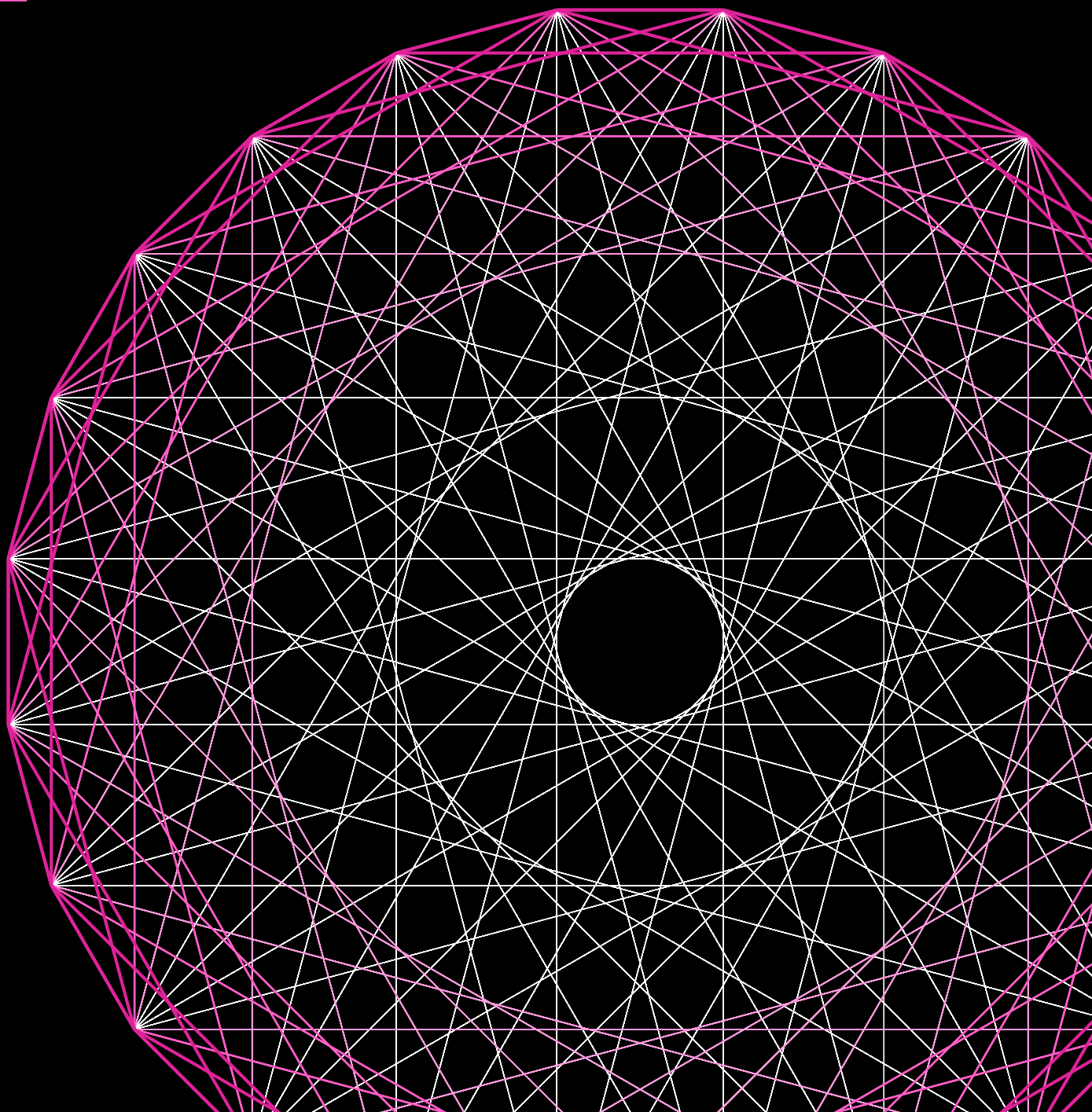

INSTRUCTIONS FOR AUTHORS: All submissions should include the name, institutional affiliation (if applicable), and postal and electronic mail addresses of each author. General inquiries may be addressed to the editorial board at `editor@mathreview.uwaterloo.ca`.

ARTICLES: *The Waterloo Mathematics Review* invites both original research and quality expository articles written for an advanced undergraduate audience on topics related to mathematics at the undergraduate level including: computer science, mathematical physics, statistics, mathematical finance, and actuarial science.

Articles may be submitted electronically or in hard copy. Electronic submissions should be in Adobe PDF format, accompanied by the relevant source files (figures, TEX files, etc.), and sent as an attachment to `submissions@mathreview.uwaterloo.ca`. Hard copies may be sent to

The Waterloo Mathematics Review
Faculty of Mathematics
University of Waterloo
200 University Ave. West
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Submissions should include an abstract and bibliography (preferably in BibTEX format). Figures must be of a production quality, and submitted in a vector format (e.g. encapsulated PostScript). The *Review* can provide assistance in vectorizing hand-drawn figures.

MATHEMATICAL ART: The *Review* invites graphics of a mathematical nature. Submissions of graphics should include both black-and-white and colour versions in production quality, along with a description of the mathematics being depicted and a few words on how the image was produced. Source code submission is not required when images are produced using a custom program, however, should an artist request it the *Review* will publish the source code on our website, to accompany the electronic version of the issue.

JOIN THE REVIEW: Are you passionate about mathematics? The *Review* is currently looking for new members for the review board. If you are an interested undergraduate at the University of Waterloo please contact the editorial board at `editor@mathreview.uwaterloo.ca`. If reviewing papers does not interest you but you would like to be involved (or you are a student at another institution and would like to join the effort) the *Review* is also looking for a webmaster and local distribution partners at other institutions. Contact the editorial board at `editor@mathreview.uwaterloo.ca` for more information.

THE COVER: The cover depicts all of the non-crossing matchings on a graph of 24 nodes that are a fixed point of at least one member of the group cyclic group generated by rotation by $\pi/12$, coloured according to the rotation of which they are a fixed point. The bars to the left illustrate $f(q) = (C_{24})_q$ evaluated around the unit circle in the complex plane. Together they provide an illustration of the cyclic sieving phenomenon, discussed in the issue. The cover was designed by Edgar A. Bering IV using a combination of custom software and *Inkscape*.

ARCHIVE: Past, present, and future issues of the *Review* are available in whole or in part on our website:

`mathreview.uwaterloo.ca`

Machine-readable citations and bibliography data from articles are also available.

SUBSCRIPTIONS: The *Review* currently does not offer subscriptions. Instead it is distributed free of charge at institutions across Canada. If you would like to see the *Review* at your school, please contact the editors.

The Waterloo Mathematics Review
Volume I, Issue 1

Winter 2011

Contents

Remarks

From the Editors

Dear Reader,

We are pleased to present the inaugural issue of *The Waterloo Mathematics Review*. The *Review* originated from a desire to inspire students by exposing them to front-line mathematics. While our graduate advisers were undergraduates here at Waterloo, they noticed a lack of scholarly material written at the undergraduate level. To ameliorate this, they proposed the creation of a new journal that would provide current undergraduates with exposure to mathematics beyond the confines of the standard curriculum. They also felt that a journal run by career academics or graduate students for this purpose would be insufficient; and so in their proposal for this new journal they suggested it be run entirely by current undergraduate students. The *Review* is that journal, and we hope that it will serve the envisioned purposes for years to come. Looking to the future, we intend to publish triennially and distribute the *Review* to universities across Canada.

For the inaugural issue we have selected articles from a wide range of mathematical areas reflecting the breadth of mathematical interest at the University of Waterloo. In this issue, you will find a discussion of diverse disciplines of mathematics from Algebraic Combinatorics to Quantum Theory. The articles vary in complexity from accessible to first year students to topics intended for senior students, though it is our hope that there is something of interest for any student of mathematics on these pages.

We would like to sincerely thank Dr. Frank Zorzitto for his guidance and support from the birth of this project. Additionally we would like to thank Dean Ian Goulden and the Faculty of Mathematics, without whom this journal would not exist, for graciously providing us with the resources to bring this idea to life. We are also grateful to the Mathematics Endowment Fund for early financial support. Finally, we would like to thank our general manager Richard Zsolt for his advice, dedication, and overall help in putting this journal together.

Yours truly,
Edgar A. Bering IV
Eeshan Wagh
Frank Ban
Editors-in-Chief
editor@mathreview.uwaterloo.ca

From the CUMC 2011 Organizing Committee

Dear mathematics students,

It is our great pleasure to invite you to the 18th Canadian Undergraduate Mathematics Conference. It will be held June 15th to 19th at Laval University, in Québec City. We encourage students who are interested in any math-related field to participate.

The CUMC is an annual gathering of undergraduate students interested in mathematics and related fields, such as physics, statistics, bioinformatics, economics and computer science. It is the largest event of its kind in North America and attracts over one hundred participants each year.

This conference is a unique opportunity to meet fellow students from across the country, encounter new areas of mathematics, hear renowned keynote speakers from a variety of disciplines and even give a talk on a topic of your choice (if you wish to). It is also a chance to explore Laval University and Québec City.

Moreover, most universities can give you funding to attend. Ask your student society or your department as soon as possible.

Here are the 8 keynote speakers attending:

- Pamela Gorkin (Complex Analysis), Bucknell University
- Aurélie Labbe (Genetic Statistics), McGill University
- Jean-Marie De Koninck (Number Theory), Laval University
- Yves Demay (Finite Element Analysis), University of Nice
- Thomas Brüstle (Algebra), Bishop's University and University of Sherbrooke
- Frédéric Gourdeau (Geometry), Laval University
- Yvan St-Aubin (Mathematical Physics), University of Montréal
- Frederick Rickey (History of Mathematics), United States Military Academy

Visit `cumc.math.ca` to learn more about it and to register.

Looking forward to seeing you at the conference,
The CUMC 2011 organizing committee
`cumc.2011@gmail.com`

From the Fields Undergraduate Network

An open call for ambassadors;

The Fields Undergraduate Network (FUN) has successfully completed its first trial run of five events. These took place between September 2010 and March 2011. We are currently undergoing a shift in governance and are looking for student ambassadors to provide continuity for the upcoming year.

These student ambassadors will represent their home university at all FUN events. They would serve as the main contacts for the organization of events, represent the interests of the math students at their home institution at meetings, and contribute ideas for the future growth of the our undergraduate mathematical community.

The tasks associated with these positions will require organizational experience. All previous ambassadors have been executives of undergraduate mathematics student societies. Length of term for these positions can be negotiated. Our ambassadors should have strong connections with their local student math societies or mathematics departments.

This call for participation is specific to a series of meetings that are being planned and will be held at the Fields Institute throughout May 2011. The exact dates are still to be decided but if you are interested and would like to receive updates on meeting dates and agenda items then please email me at `richard.cerezo@utoronto.ca` to be added to the mailing list.

To view our first set of events, please visit our webpage:
`http://www.fields.utoronto.ca/programs/outreach/10-11/undergradnet/index.html`

Thank you,
Richard Cerezo
FUN, Founder

Cyclic Sieving Phenomenon

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Abstract: The cyclic sieving phenomenon is an interesting phenomenon with connections to enumeration and representation theory. We will study the canonical example of multisets and present two proofs that illustrate these connections. We conclude by looking at a few other examples of the cyclic sieving phenomenon. Much of this paper is based on Sagan's survey.

1 Introduction

1.1 Definition of the CSP

Suppose that we have a cyclic group C acting on a set X. In combinatorics, it is natural to ask the number of fixed points: $|X^g| = |\{x \in X : gx = x\}|$. In their 2004 paper Reiner, Stanton, and White describe a phenomenon where one polynomial encodes all numbers of fixed elements by cyclic actions [**?**]. This is called the cyclic sieving phenomenon.

Definition 1.1. Let $C = \{1, c, c^2, \cdots, c^{n-1}\}$ be a finite cyclic group acting on a finite set X. Let $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$ be a root of unity of order n and let $f(q)$ be a polynomial with non-negative integer coefficients. We say that the triple $(X, C, f(q))$ *exhibits the cyclic sieving phenomenon (CSP)* if for any non-negative integer d, we have that the fixed point set cardinality $|X^{c^d}|$ is equal to the polynomial evaluated at $f(\zeta^d)$.

Remark 1.1: A trivial remark one needs to make is that $f(1)$ is equal to the number of elements in X.

Remark 1.2: Having an action by $C = \{1, c\}$ is equivalent to having an involution on X (i.e. a bijection that is its own inverse). Then the CSP corresponds to the result given by Stembridge in 1993 [**?**]. In fact, Stembridge's result was the motivation of the CSP.

Remark 1.3: It is easy to see that $f(q)$ is unique up to the polynomial $x^n - 1$. If $(X, C, f_1(q))$ and $(X, C, f_2(q))$ exhibit the CSP, then $f_1(q) - f_2(q)$ is a polynomial that has $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$ as roots, so $x^n - 1$ divides $f_1(q) - f_2(q)$.

Remark 1.4: If $f(q) = \sum_{k=0}^{n-1} a_k q^k$ where a_k is the number of C-orbits in X with stabilizer order (i.e. the size of C divided by the size of the orbit) dividing k, then

$$f(q) = \sum_{k=0}^{n-1} a_k q^k = \sum_{\text{orbit } O} 1 + q^{|C|/|O|} + q^{2|C|/|O|} + \cdots + q^{(|O|-1)|C|/|O|}.$$

If $f(q)$ is evaluated at $q = \zeta^d$, $1 + q^{|C|/|O|} + q^{2|C|/|O|} + \cdots + q^{(|O|-1)|C|/|O|}$ is $|O|$ if the order of ζ^d divides $|C|/|O|$ and 0 otherwise. Therefore, $f(\zeta^d) = |X^{c^d}|$ and $(X, C, f(q))$ exhibits the CSP. Combined with the above remark, we now have existence and uniqueness of $f(q)$.

Note that in many situations, these polynomials are naturally associated to the combinatorial structure of the set X; $f(q)$ is often a generating function associated with X.

Before we present an example of the CSP, let us quickly introduce couple tools we will use.

1.2 PREREQUISITE: Q-ANALOGS

Define the the *q-analog of the number* n as $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Let $[n]_q! = [1]_q[2]_q \cdots [n]_q$ be the *q-analog of* $n!$ and $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ be the *q-analog of* $\binom{n}{k}$. These q-analogs are polynomials in q and tend to our ordinary numbers, factorials, and binomial coefficients as q approaches 1.

The q-analog of a binomial coefficient adds extra information to each object the binomial coefficient is counting. For example, if $\binom{n}{k}$ counts the number of monotonic lattice path inside the $(n-k) \times k$ box, then its q-analog also takes the number of boxes that are above the path into account.

We remarked that $f(1) = |X|$ when $(X, C, f(q))$ exhibits the CSP. In many cases, f turns out to be the q-analog of the number of elements in X.

1.3 PREREQUISITE: REPRESENTATION THEORY

Let V be a complex vector space. The group of invertible linear maps from V to itself is denoted as $GL(V)$. A group homomorphism $[\cdot] : G \to GL(V)$ is called a *representation*. Representations provide a way to study group theory using linear algebra.

In this article, we will use representations when G acts on V. Then there is a natural choice of representation: $[g] : v \mapsto gv$.

Given a choice of basis B on V, $[g]$ can be written in a matrix form. Define the *character* of a representation to be $\chi : G \to \mathbb{C}$ such that $\chi(g) = \mathrm{tr}[g]$. Since the trace is independent of the choice of basis, $\chi(g)$ is well-defined.

Example 1.1. If $V = \mathbb{C}^3$ and $g = (1\,2) \in S_3$ acts on V by switching the first two components, then the matrix form of g in the standard basis is

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\chi(g)$ is 1.

2 CANONICAL EXAMPLE

2.1 EXAMPLE OF MULTISETS

The canonical example of a combinatorial structure exhibiting the CSP is the example of multisets. Let positive integers n and k be fixed. A k-multiset of $[n] = \{1, 2, 3, \cdots, n\}$ is an unordered family of k elements of $[n]$ where repetitions are allowed. For example, the set of 3-multisets of [3] is $\{111, 222, 333, 112, 113, 221, 223, 331, 332, 123\}$. Let X be the set of k-multisets of $[n]$.

Let C be the cyclic subgroup of S_n generated by the cycle $c = (1\,2\,3\,\cdots\,n)$. C acts naturally on X: if $M = m_1 m_2 \cdots m_k$ is a multiset, then $gM = g(m_1)g(m_2)\cdots g(m_k)$ where $g \in C$. For example, $(1\,2\,3)223 = 331$.

There are $\binom{n+k-1}{k}$ such multisets. For our polynomial, we take the q-analog. Define $f(q)$ as $\binom{n+k-1}{k}_q$.

Theorem 2.1. $(X, C, f(q))$ defined as above exhibits the cyclic sieving phenomenon.

Example 2.1. Let $n = 3$ and $k = 3$. $f(q)$ is

$$\binom{3+3-1}{3}_q = \frac{(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)}{1(1+q)} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

$|X^{id}| = |X|$ is 10. $f(\zeta^0) = f(1) = 10$, so $|X^{id}| = f(\zeta^0)$. $|X^{(1\,2\,3)}|$ is 1 since 123 is the only fixed multiset. $f(\zeta^1) = 1 + \zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4 + \zeta^5 + \zeta^6 = 4 + 3\zeta + 3\zeta^2 = 1$ since $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$. Therefore, $|X^{(1\,2\,3)}| = f(\zeta)$.

2.2 Proof by direct evaluation

One can prove the above theorem by simply evaluating both $|X^{c^d}|$ and $f(\zeta^d)$ explicitly. The following two lemmas show the evaluation of these two sides.

Lemma 2.1. Let o be the order of c^d. Then $|X^{c^d}| = \binom{n/o+k/o-1}{k/o}$ if $o|k$ and 0 otherwise.

Proof. If $g = c_1c_2\cdots c_t \in C$ is the cycle decomposition of $g \in C$, then $x \in X$ is fixed under the action by g if and only if x can be written as disjoint union of the cycles c_i with repetition allowed. For example, if $g = c^2 = (1\,2\,3\,4\,5\,6)^2 = (1\,3\,5)(2\,4\,6)$, then the multisets fixed by g must have the form $1^a3^a5^a2^b4^b6^b$. This claim is easy to check.

c^d decomposes into n/o cycles of size o, so no k-multiset is fixed if o does not divide k. If it does, then one must choose k/o cycles with repetition allowed to form a multiset of size k that is fixed under the action by c^d. Therefore, $|X^{c^d}| = \binom{n/o+k/o-1}{k/o}$. □

Lemma 2.2. Let o be the order of ζ^d. Then $f(\zeta^d) = \binom{n/o+k/o-1}{k/o}$ if $o|k$ and 0 otherwise.

Proof. Note that for any non-negative integer a and b, $\frac{[ao+k]_q}{[bo+k]_q}$ is $\frac{1+q+q^2+\cdots+q^{ao+k-1}}{1+q+q^2+\cdots+q^{bo+k-1}}$. When $q = \zeta^d$, this is equal to $\frac{1+q^o+q^{2o}+\cdots+q^{(a-1)o}}{1+q^o+q^{2o}+\cdots+q^{(b-1)o}} = \frac{a}{b}$ if $k = 0$. Otherwise, it is $\frac{1+q+q^2+\cdots+q^{k-1}}{1+q+q^2+\cdots+q^{k-1}} = 1$.

If o does not divide k, then $f(q) = \binom{n+k-1}{k}_q$ has more $[o]_q$ factors in the numerator than the denominator, so $f(\zeta^d) = 0$. Otherwise,

$$\begin{aligned} f(\zeta^d) = \binom{n+k-1}{k}_{\zeta^d} &= \frac{[n]_{\zeta^d}}{[k]_{\zeta^d}}\frac{[n+1]_{\zeta^d}}{[1]_{\zeta^d}}\cdots\frac{[n+k-1]_{\zeta^d}}{[k-1]_{\zeta^d}} \\ &= \frac{n}{k}\cdot 1\cdot 1\cdots 1\cdot\frac{n+o}{o}\cdot 1\cdot 1\cdots 1\cdot\frac{n+2o}{2o}\cdots \\ &= \frac{n/o}{k/o}\cdot\frac{n/o+1}{1}\cdot\frac{n/o+2}{2}\cdots \\ &= \binom{n/o+k/o-1}{k/o}, \end{aligned}$$

which is what we wanted. □

By these two lemmas, we conclude that $|X^{c^d}| = f(\zeta^d)$. Therefore, $(X, C, f(q))$ exhibits the cyclic sieving phenomenon.

2.3 Proof by representation theory

The previous proof is elementary, but it does not tell us much about why the equality holds. We now present an another proof that uses representation theory and provides more insight into our situation.

The main idea of the proof is to evaluate a character using two different bases. In one basis, the character will be the number of fixed elements, which is combinatorial information. In another basis, the character will be algebraically realized as a polynomial.

Proof of Theorem **??**. Given a set $S = \{s_1, s_2, \cdots, s_n\}$, we define a complex vector space

$$\mathbb{C}S = \{c_1s_1 + c_2s_2 + \cdots + c_ns_n | c_i \in \mathbb{C}\}.$$

An element $g \in S_n$ acts naturally on $\mathbb{C}[n]$: $g(c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n}) = c_1g(\mathbf{1}) + c_2g(\mathbf{2}) + \cdots + c_ng(\mathbf{n})$. Define $\mathrm{Sym}_k(n)$ as $\mathbb{C}X$ where X is the set of multisets of $[n]$ of size k.

Clearly, X is the standard basis of $\mathrm{Sym}_k(n)$. A diagonal entry of $[g]_X$ is 1 if the multiset $M \in X$ is fixed by g and 0 otherwise. Therefore, $\chi(g^d) = \mathrm{tr}[g^d]_X = |X^{g^d}|$.

Example 2.2. If $n = 3$ and $k = 3$ as before and $g = c^1 = (1\,2\,3)$, then

$$\begin{array}{llllll} g(111) = 222, & g(222) = 333, & g(333) = 111, & g(112) = 223, & g(113) = 221, \\ g(221) = 332, & g(223) = 331, & g(331) = 112, & g(332) = 113, & g(123) = 123. \end{array}$$

So $[g]_{\{111,222,333,112,113,221,223,331,332,123\}}$ is equal to

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Only one diagonal entry is 1 because only 123 is fixed under the action g. Therefore, $\chi(g) = 1$.

We want another way of evaluating this character $\chi(g)$, so that it yields $f(\zeta^d)$. Let $c = (1\,2\,3\,\cdots\,n) \in S_n$. c^n is the identity in S_n, so $[c]^n$ is also the identity in $GL(\mathbb{C}[n])$. Thus, the characteristic polynomial of $[c]$ is $x^n - 1$, which has n distinct roots: $x_1 = 1, x_2 = \zeta, x_3 = \zeta^2, \cdots, x_n = \zeta^{n-1}$. Therefore, there must be a basis $B = \{b_1, b_2, \cdots, b_n\}$ of $\mathbb{C}[n]$ such that $[c] \in GL(\mathbb{C}[n])$ is diagonalized with respect to B and has $x_1, x_2 \ldots, x_n$ on the diagonal. Note $[c^d]_B = \mathrm{diag}(x_1^d, x_2^d, \cdots, x_n^d)$. Let B' be the set of k-multisets of B, then B' is another basis for $\mathrm{Sym}_k(n)$ (this is a property of $\mathrm{Sym}_k(n)$).

We now evaluate the character of $g = c^d$ with the basis B'.

$$g(b_{i_1} b_{i_2} \cdots b_{i_k}) = g(b_{i_1}) g(b_{i_2}) \cdots g(b_{i_k}) = x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d b_{i_1} b_{i_2} \cdots b_{i_k}.$$

So it follows that the diagonal entries of $[g]_{B'}$ are $x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d$ and the trace of $[g]$ is $\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1}^d x_{i_2}^d \cdots x_{i_k}^d$.

The polynomial $\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} y_{i_1}^d y_{i_2}^d \cdots y_{i_k}^d$ is called the *complete homogeneous symmetric polynomial* in n variables of degree k and it is denoted by $h_k(y_1^d, y_2^d, \cdots, y_n^d)$.

It remains to show that $h_k(1, q, q^2, \cdots, q^{n-1})$ is equal to $\binom{n+k-1}{k}_q$. One can check that

$$h_k(1, q, q^2, \cdots, q^{n-1}) = h_k(1, q, q^2, \cdots, q^{n-2}) + q^{n-1} h_{k-1}(1, q, q^2, \cdots, q^{n-1})$$

and

$$\binom{n+k-1}{k}_q = \binom{n+k-2}{k}_q + q^{n-1} \binom{n+k-2}{k-1}_q.$$

Since the recursions are identical, the equality follows by induction. □

Example 2.3. Again picking $n = 3$, $k = 3$, and $g = c^1 = (1\,2\,3)$,

$$\begin{array}{lll} g(b_1b_1b_1) = x_1^3 b_1b_1b_1, & g(b_2b_2b_2) = x_2^3 b_2b_2b_2, & g(b_3b_3b_3) = x_3^3 b_3b_3b_3, \\ g(b_1b_1b_2) = x_1^2 x_2 b_1b_1b_2, & g(b_1b_1b_3) = x_1^2 x_3 b_1b_1b_3, & g(b_2b_2b_1) = x_2^2 x_1 b_2b_2b_1, \\ g(b_2b_2b_3) = x_2^2 x_3 b_2b_2b_3, & g(b_3b_3b_1) = x_3^2 x_1 b_3b_3b_1, & g(b_3b_3b_2) = x_3^2 x_2 b_3b_3b_2, \\ g(b_1b_2b_3) = x_1x_2x_3 b_1b_2b_3. & & \end{array}.$$

So $[g]_{\{b_1b_1b_1, b_2b_2b_2, b_3b_3b_3, b_1b_1b_2, b_1b_1b_3, b_2b_2b_1, b_2b_2b_3, b_3b_3b_1, b_3b_3b_2, b_1b_2b_3\}}$ is equal to

$$\begin{bmatrix} x_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1^2x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2^2x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2^2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3^2x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1x_2x_3 \end{bmatrix}.$$

Therefore, $\chi(g) = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3 = h_3(x_1, x_2, x_3)$.

3 Other Examples

3.1 Subsets of [n]

We have seen that multisets exhibit the CSP. Subsets similarly exhibit the CSP.

Theorem 3.1.

$$\left(\{S \subseteq [n] : |S| = k\}, \langle (1\,2\,\cdots\,n)\rangle, \binom{n}{k}_q\right)$$

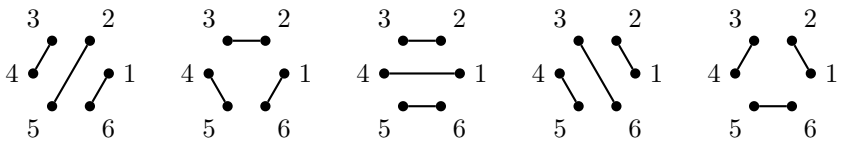
exhibits the cyclic sieving phenomenon.

Proof. The proof is similar to the proof of the multiset case, but it is more technical. □

3.2 Catalan numbers

A *matching* is a graph G with vertex set $[2n]$ and n edges, no two of which share a common vertex. The matching is *non-crossing* if it does not contain a pair of edges ab and cd such that $a < c < b < d$. It is equivalent to say that if the vertices are arranged in order around a circle, no two edges intersect. The nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, is the number of non-crossing matchings of size $2n$.

Example 3.1. $C_3 = 5$.



Rotation by π/n is an action on non-crossing matchings.

Theorem 3.2. Let X be the set of non-crossing matchings of size $2n$ and let R be a rotation by π/n. Then

$$\left(X, \langle R\rangle, \frac{1}{[2n+1]_q}\binom{2n}{n}_q\right)$$

exhibits the cyclic sieving phenomenon.

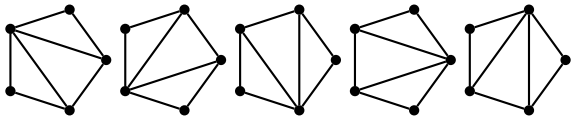
For readers who are familiar with Tableaux theory: this result is a special case of a CSP result on rectangular tableaux, which says that if $X = \{\text{standard Young tableaux of shape } n \times d\}$ then

$$(X, \langle \text{jeu-de-taquin promotion}\rangle, \text{q-analog of hook length formula})$$

exhibits the CSP. The Catalan case arises when $d = 2$. This result can be found in Rhoades' paper [**?**].

There are other interesting cyclic actions one can consider on Catalan combinatorics. We end by showing one other cyclic action that exhibits the CSP. It is well known that the number of triangulations of a regular $n+2$-gon is C_n.

Example 3.2. $C_3 = 5$



Rotation by $2\pi/(n+2)$ is an action on triangulations.

Theorem 3.3. Let X be the set of triangulations of a regular $n+2$-gon and R be a rotation by $2\pi/(n+2)$. Then

$$\left(X, \langle R\rangle, \frac{1}{[2n+1]_q}\binom{2n}{n}_q\right)$$

exhibits the cyclic sieving phenomenon.

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Iterative Methods for Computing Eigenvalues and Eigenvectors

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Abstract: We examine some numerical iterative methods for computing the eigenvalues and eigenvectors of real matrices. The five methods examined here range from the simple power iteration method to the more complicated QR iteration method. The derivations, procedure, and advantages of each method are briefly discussed.

1 Introduction

Eigenvalues and eigenvectors play an important part in the applications of linear algebra. The naive method of finding the eigenvalues of a matrix involves finding the roots of the characteristic polynomial of the matrix. In industrial sized matrices, however, this method is not feasible, and the eigenvalues must be obtained by other means. Fortunately, there exist several other techniques for finding eigenvalues and eigenvectors of a matrix, some of which fall under the realm of iterative methods. These methods work by repeatedly refining approximations to the eigenvectors or eigenvalues, and can be terminated whenever the approximations reach a suitable degree of accuracy. Iterative methods form the basis of much of modern day eigenvalue computation.

In this paper, we outline five such iterative methods, and summarize their derivations, procedures, and advantages. The methods to be examined are the power iteration method, the shifted inverse iteration method, the Rayleigh quotient method, the simultaneous iteration method, and the QR method. This paper is meant to be a survey of existing algorithms for the eigenvalue computation problem.

Section 2 of this paper provides a brief review of some of the linear algebra background required to understand the concepts that are discussed. In Section 3, the iterative methods are each presented, in order of complexity, and are studied in brief detail. Finally, in Section 4, we provide some concluding remarks and mention some of the additional algorithm refinements that are used in practice.

For the purposes of this paper, we restrict our attention to real-valued, square matrices with a full set of real eigenvalues.

2 Linear Algebra Review

We begin by reviewing some basic definitions from linear algebra. It is assumed that the reader is comfortable with the notions of matrix and vector multiplication.

Definition 2.1. Let $A \in \mathbb{R}^{n \times n}$. A nonzero vector $x \in \mathbb{R}^n$ is called an *eigenvector* of A with corresponding *eigenvalue* $\lambda \in \mathbb{C}$ if $Ax = \lambda x$.

Note that eigenvectors of a matrix are precisely the vectors in $\mathbb{R}^n$ whose direction is preserved when multiplied with the matrix. Although eigenvalues may be not be real in general, we will focus on matrices whose eigenvalues are all real numbers. This is true in particular if the matrix is symmetric; some of the methods we detail below only work for symmetric matrices.

It is often necessary to compute the eigenvalues of a matrix. The most immediate method for doing so involves finding the roots of characteristic polynomials.

Definition 2.2. The *characteristic polynomial* of A, denoted $P_A(x)$ for $x \in \mathbb{R}$, is the degree n polynomial defined by

$$P_A(x) = \det(xI - A).$$

It is straightforward to see that the roots of the characteristic polynomial of a matrix are exactly the eigenvalues of the matrix, since the matrix $\lambda I - A$ is singular precisely when λ is an eigenvalue of A. It follows that computation of eigenvalues can be reduced to finding the roots of polynomials. Unfortunately, solving polynomials is generally a difficult problem, as there is no closed formula for solving polynomial equations of degree 5 or higher. The only way to proceed is to employ numerical techniques to solve these equations.

We have just seen that eigenvalues may be found by solving polynomial equations. The converse is also true. Given any monic polynomial,

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0,$$

we can construct the companion matrix

$$\begin{bmatrix} 0 & & & \cdots & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & & \ddots & & \vdots \\ & & & 1 & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}.$$

It can be seen that the characteristic polynomial for the companion matrix is exactly the polynomial $f(z)$. Thus the problem of computing the roots of a polynomial equation reduces to finding the eigenvalues of a corresponding matrix. Since polynomials in general cannot be solved exactly, it follows that there is no method that will produce exact eigenvalues for a general matrix.

However, there do exist methods for computing eigenvalues and eigenvectors that do not rely upon solving the characteristic polynomial. In this paper, we look at some iterative techniques used for tackling this problem. These are methods that, when given some initial approximations, produce sequences of scalars or vectors that converge towards the desired eigenvalues or eigenvectors. On the other hand, we can make the notion of convergence of matrices precise as follows.

Definition 2.3. Let $A^{(1)}, A^{(2)}, A^{(3)}, \ldots$ be a sequence of matrices in $\mathbb{R}^{m\times n}$. We say that the sequence of matrices *converges* to a matrix $A \in \mathbb{R}^{m\times n}$ if the sequence $A_{i,j}^{(k)}$ of real numbers converges to $A_{i,j}$ for every pair $1 \le i \le m$, $1 \le j \le n$, as k approaches infinity. That is, a sequence of matrices converges if the sequences given by each entry of the matrix all converge.

Later in this paper, it will be necessary to use what is known as the QR decomposition of a matrix.

Definition 2.4. The *QR decomposition* of a matrix A is the representation of A as a product

$$A = QR,$$

where Q is an orthogonal matrix and R is an upper triangular matrix with positive diagonal entries.

Recall that an *orthogonal* matrix U satisfies $U^TU = I$. Importantly, the columns of Q are orthogonal vectors, and span the same space as the columns of A. It is a fact that any matrix A has a QR decomposition $A = QR$, which is unique when A has full rank.

Geometrically, the QR factorization means that if the columns of A form the basis of a vector space, then there is an orthonormal basis for that vector space. This orthonormal basis would would form the columns of Q, and the conversion matrix for this change of basis is the upper triangular matrix R. The

methods for obtaining a QR decomposition of a matrix has been well studied and is a computationally feasible task.[1]

At this point, we turn our attention to the iterative methods themselves.

3 Description of the Iterative Methods

The iterative methods in this section work by repeatedly refining estimates of eigenvalues of a matrix, using a function called the Rayleigh quotient.

Definition 3.1. Let $A \in \mathbb{R}^{n\times n}$. Then the *Rayleigh quotient* of a nonzero vector $x \in \mathbb{R}^n$ is

$$r(x) = \frac{x^T Ax}{x^T x}.$$

Note that if x is an eigenvector for A with corresponding eigenvalue λ, then the Rayleigh quotient for x is

$$r(x) = \frac{x^T Ax}{x^T x} = \frac{\lambda x^T x}{x^T x} = \lambda,$$

which is exactly the corresponding eigenvalue.

In fact, given any nonzero $x \in \mathbb{R}^n$, the Rayleigh quotient $r(x)$ is the value that minimizes the function $f(\alpha) = \|\alpha x - Ax\|_2$ over all real numbers α, which measures the error incurred if x is assumed to be an eigenvector of A. Thus, if we treat x as an estimate for an eigenvector of A, then $r(x)$ can be seen as the best estimate for the corresponding eigenvalue of A, since it minimizes this error value.

We are now ready to consider the first technique for iterative eigenvalue computation.

3.1 Power iteration

Let $A \in \mathbb{R}^{n\times n}$. Recall that if q is an eigenvector for A with eigenvalue λ, then $Aq = \lambda q$, and in general, $A^k q = \lambda^k q$ for all $k \in \mathbb{N}$. This observation is the foundation of the power iteration method.

Suppose that the set $\{q_i\}$ of unit eigenvectors of A forms a basis of $\mathbb{R}^n$, and has corresponding real eigenvalues $\{\lambda_i\}$ such that $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. Let $v^{(0)}$ be an approximation to an eigenvector of A, with $\left\|v^{(0)}\right\| = 1$. (We use the term "approximation" in this situation quite loosely, allowing it to refer to any quantity that is believed to be "reasonably close" to the correct value.[2]) We can write $v^{(0)}$ as a linear combination of the eigenvectors of A; for some $c_1, \ldots, c_n \in \mathbb{R}$ we have that

$$v^{(0)} = c_1 q_1 + \ldots + c_n q_n,$$

and we will assume for now that $c_1 \neq 0$.

Now

$$Av^{(0)} = c_1\lambda_1 q_1 + c_2\lambda_2 q_2 + \ldots + c_n\lambda_n q_n$$

and so

$$\begin{aligned} A^k v^{(0)} &= c_1\lambda_1^k q_1 + c_2\lambda_2^k q_2 + \ldots + c_n\lambda_n^k q_n \\ &= \lambda_1^k \left(c_1 q_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k q_2 + \ldots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k q_n \right). \end{aligned}$$

[1] The simplest method for computing a QR factorization of a matrix A is to apply the Gram-Schmidt algorithm on the columns of A. A student of linear algebra may be horrified at the prospect of carrying out this tedious procedure once, let alone once *per iteration* of an iterative method, but recall that when working with matrices of thousands of rows or more, all computations are done electronically. Furthermore, the methods used to calculate a QR decomposition are usually more complicated than Gram-Schmidt; for instance, a common technique is to apply a series of Householder transformations [GL87].

[2] These approximations are always made when the correct value is unknown (indeed, they are made in an attempt to *determine* the correct value); we therefore do not require any particularly demanding conditions on how "close" the approximation must be to being correct.

Since the eigenvalues are assumed to be real, distinct, and ordered by decreasing magnitude, it follows that for all $i = 2, \ldots, n$,

$$\lim_{k \to \infty} \left(\frac{\lambda_i}{\lambda_1} \right)^k = 0.$$

So, as k increases, $A^k v^{(0)}$ approaches $c_1 \lambda_1^k q_1$, and thus for large values of k,

$$q_1 \approx \frac{A^k v^{(0)}}{\left\| A^k v^{(0)} \right\|}.$$

The method of power iteration can then be stated as follows:

```
Pick a starting vector v^(0) with ||v^(0)|| = 1
For k = 1, 2, ...
    Let w = A v^(k-1)
    Let v^(k) = w / ||w||
```

In each iteration, $v^{(k)}$ gets closer and closer to the eigenvector q_1. The algorithm may be terminated at any point with a reasonable approximation to the eigenvector; the eigenvalue estimate can be found by applying the Rayleigh quotient to the resulting $v^{(k)}$.

The power iteration method is simple and elegant, but suffers some major drawbacks. The method only returns a single eigenvector estimate, and it is always the one corresponding to the eigenvalue of largest magnitude. In addition, convergence is only guaranteed if the eigenvalues are distinct—in particular, the two eigenvalues of largest absolute value must have distinct magnitudes. The rate of convergence primarily depends upon the ratio of these magnitudes, so if the two largest eigenvalues have similar sizes, then the convergence will be slow.

In spite of its drawbacks, the power method is still used in some applications, since it works well on large, sparse matrices when only a single eigenvector is needed. However, there are other methods that overcome the difficulties of the power iteration method.

3.2 Inverse iteration

The inverse iteration method is a natural generalization of the power iteration method.

If A is an invertible matrix with real, nonzero eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, then the eigenvalues of A^{-1} are $\{1/\lambda_1, \ldots, 1/\lambda_n\}$. Thus if $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$, then $|1/\lambda_1| < |1/\lambda_2| < \ldots < |1/\lambda_n|$, and so by applying the power method iteration on A^{-1}, we can obtain the eigenvector q_n and eigenvalue λ_n.

This gives a way to find the eigenvalue of smallest magnitude, assuming that A^{-1} is known. In general, though, the inverse matrix is not given, and calculating it is a computationally expensive operation. However, computing $x = A^{-1}b$ is equivalent to solving the system $Ax = b$ for x given b, and this operation can be efficiently performed. Fortunately, this is all that is required for the inverse iteration method, which we can now state as follows:

```
Pick a starting vector v^(0) with ||v^(0)|| = 1
For k = 1, 2, ...
    Solve A w = v^(k-1) for w
    Let v^(k) = w / ||w||
```

The advantage of inverse iteration is that it can be easily adapted to find *any* eigenvalue of the matrix A, instead of just the extreme ones. Observe that for any $\mu \in \mathbb{R}$, the matrix $B = A - \mu I$ has eigenvalues $\{\lambda_1 - \mu, \ldots, \lambda_n - \mu\}$. In particular, by choosing μ to be close to an eigenvalue λ_j of A, we can ensure that $\lambda_j - \mu$ is the eigenvalue of B of smallest magnitude. Then by applying inverse iteration on B, an approximation to q_j and λ_j can be obtained.

This version of the algorithm, known as inverse iteration with shift, can be summarized as follows:

```
Pick some μ close to the desired eigenvalue
Pick a starting vector v^(0) with ||v^(0)|| = 1
For k = 1, 2, ...
    Solve (A − μI)w = v^(k−1) for w
    Let v^(k) = w/||w||
```

The inverse iteration with shift method allows the computation of any eigenvalue of the matrix. However, in order to compute a particular eigenvalue, you must have some initial approximation of it to start the iteration. In cases where an eigenvalue estimate is given, the inverse iteration with shift method works well.

3.3 Rayleigh quotient iteration

The inverse iteration method can be improved if we drop the restriction that the shift value remains constant throughout the iterations.

Each iteration of the shifted inverse iteration method returns an approximate eigenvector, given an estimate of an eigenvalue. The Rayleigh quotient, on the other hand, produces an approximate eigenvalue when given an estimated eigenvector. By combining these two operations, we get a new variation of the inverse shift algorithm, where the shift value is recomputed during each iteration to become the Rayleigh quotient of the current eigenvector estimate. This method, called the Rayleigh quotient iteration method (or simply the RQI method), is as follows:

```
Pick a starting vector v^(0) with ||v^(0)|| = 1
Let λ^(0) = r(v^(0)) = (v^(0))^T A(v^(0))
For k = 1, 2, ...
    Solve (A − λ^(k−1) I)w = v^(k−1) for w
    Let v^(k) = w/||w||
    Let λ^(k) = r(v^(k)) = (v^(k))^T A(v^(k))
```

In this method, we no longer need to have an initial eigenvalue estimate supplied; all that is required is an initial vector $v^{(0)}$. The eigenvector produced depends on the initial vector chosen. Note that since each vector $v^{(k)}$ is a unit vector, we have $\left(v^{(k)}\right)^T v^{(k)} = 1$, simplifying the expression for $r\left(v^{(k)}\right)$.

The main advantage of the RQI method is that it converges to an eigenvector very quickly, since the approximations to both the eigenvalue and the eigenvector are improved during each iteration. Thus far, we have not given any quantitative discussion on the speed at which an iteration sequence converges to the limit, since these technical details are not the purpose of this paper. However, it is worth mentioning here that the convergence rate of the RQI method is said to be *cubic*, which means that the number of correct digits in the approximation triples during each iteration [Ost59]. In contrast, the other algorithms described in this paper all have the much slower *linear* rate of convergence.[3]

One very significant disadvantage of the RQI method is that it does not always work in the general case. The method is only guaranteed to converge when the matrix A is both real and symmetric, and is known to fail in the cases where the matrix is not symmetric.

[3]In further contrast, the well known Newton's method for rapidly finding roots of differentiable functions has *quadratic* convergence. Generally, the study of algorithm convergence rates is extremely technical, and the author feels no regret in omitting further details from this paper.

3.4 SIMULTANEOUS ITERATION

The methods discussed so far are only capable of computing a single eigenvalue at a time. In order to compute different eigenvalues of a matrix, the methods must be reapplied several times, each time with different initial conditions. We now describe a method that is capable, in some situations, of producing all of the eigenvalues of a matrix at once.

Once again, our starting point is the basic power iteration method. Let A be a real, symmetric, full rank matrix; in particular, A has real eigenvalues and a complete set of orthogonal eigenvectors. Recall that given the starting vector $v^{(0)}$, we can write it as a linear combination of eigenvectors $\{q_i\}$ of A:

$$v^{(0)} = c_1 q_1 + \ldots + c_n q_n.$$

Note, however, that the power iteration method only looks at eigenvectors that are nontrivial components in this linear combination; that is, only the eigenvectors that are not orthogonal to $v^{(0)}$ have a chance of being found by the power iteration method. This suggests that by applying the power iteration method to several different starting vectors, each orthogonal to all of the others, there is a possibility of finding different eigenvalues.

With this idea in mind, we may take the following approach. Begin with a basis of n linearly independent vectors $\{v_1^{(0)} \ldots v_n^{(0)}\}$ of $\mathbb{R}^n$, arranged in the matrix

$$V^{(0)} = \left[v_1^{(0)} \middle| \cdots \middle| v_n^{(0)} \right].$$

Let

$$V^{(k)} = A^k V^{(0)} = \left[v_1^{(k)} \middle| \cdots \middle| v_n^{(k)} \right],$$

which effectively applies the power iteration method to all of the vectors $\{v_1^{(0)} \ldots v_n^{(0)}\}$ at once. We thus expect that as $k \to \infty$, the columns of $V^{(k)}$ become scaled copies of q_1, the unit eigenvector corresponding to the eigenvalue of largest magnitude. (Note that the columns are not necessarily unit vectors themselves, since we did not normalize the vectors in this version of the algorithm).

So far, we have not found anything useful or new. We have obtained n eigenvectors, but they are possibly all in the same direction. The main development occurs when we decide to orthonormalize the columns of $V^{(k)}$ at each iteration. In the original power iteration method, the obtained eigenvector estimate was normalized in each iteration. In this multi-vector version, the analogue is to obtain an orthonormal set of eigenvector estimates during each iteration, forcing the eigenvector approximations to be orthogonal at all times. This is done by using the QR decomposition of $V^{(k)}$.

In each step of this iteration method, we obtain a new matrix, W, by multiplying A by the current eigenvector approximation matrix, $V^{(k-1)}$. We can then extract the orthonormal column vectors of Q from the QR decomposition of W, thus ensuring that the eigenvector approximations remain an orthogonal basis of unit vectors. This process is then repeated as desired. The algorithm, known as the simultaneous iteration method [Rut69], can be written as follows:

Pick a starting basis $\{v_1^{(0)}, \ldots, v_n^{(0)}\}$ of $\mathbb{R}^n$.
Build the matrix $V = \left[v_1^{(0)} \middle| \cdots \middle| v_n^{(0)} \right]$
Obtain the factors $Q^{(0)}R^{(0)} = V^{(0)}$
For $k = 1, 2, \ldots$
    Let $W = AQ^{(k-1)}$
    Obtain the factors $Q^{(k)}R^{(k)} = W$

It is a fact that if the matrix A has n orthogonal eigenvectors $q_1, \ldots, q_n$ with corresponding real eigenvalues $|\lambda_1| > \ldots > |\lambda_n|$, and if the leading principal submatrices of the product $[q_1| \cdots |q_n]^T V^{(0)}$ are nonsingular, then the columns of $Q^{(k)}$ will converge towards a basis of eigenvectors of A. (Recall that the

leading principal submatrices of the matrix B are the top left square submatrices of B.) So, at last, here is a method that, under some hypotheses, computes all of the eigenvectors of the matrix A at once.

However, this method is generally not used in practice. It turns out that there is a more elegant form of this algorithm, which we examine now.

3.5 The QR method

The QR method for computing eigenvalues and eigenvectors [QRA], like the simultaneous iteration method, allows the computation of all eigenvalues and eigenvectors of a real, symmetric, full rank matrix at once. Based upon the matrix decomposition from which it earned its name, the simplest form of the QR iteration algorithm may be written as follows:

```
Let A^(0) = A
For k = 1, 2, ...
    Obtain the factors Q^(k) R^(k) = A^(k-1)
    Let A^(k) = R^(k) Q^(k)
```

Simply put, in each iteration, we take the QR decomposition of the current matrix $A^{(k-1)}$, and multiply the factors Q and R in the *reverse* order to obtain the new matrix $A^{(k)}$.

It is surprising that this non-intuitive process would converge to anything useful, let alone a full set of eigenvectors and eigenvalues of A. However, it turns out that this algorithm can, in some sense, be seen to be equivalent to the simultaneous iteration method. The simultaneous iteration method of the previous section can be written as follows:

```
Let Q̄^(0) = I
For k = 1, 2, ...
    Let W = A Q̄^(k-1)
    Obtain the factors Q̄^(k) R^(k) = W
    Let A^(k) = (Q̄^(k))^T A Q̄^(k)
    Let R̄^(k) = R^(k) R^(k-1) ... R^(1)
```

Here we have renamed the $Q^{(k)}$ matrices of the previous section as $\overline{Q}^{(k)}$, and have introduced the matrices $A^{(k)}$ and $\overline{R}^{(k)}$ which do not affect the correctness of the algorithm. The QR method of this section can be rewritten in the following way:

```
Let A^(0) = A
For k = 1, 2, ...
    Obtain the factors Q^(k) R^(k) = A^(k-1)
    Let A^(k) = R^(k) Q^(k)
    Let Q̄^(k) = Q^(1) Q^(2) ... Q^(k)
    Let R̄^(k) = R^(k) R^(k-1) ... R^(1)
```

Again, the introduction of the matrices $\overline{Q}^{(k)}$ and $\overline{R}^{(k)}$ do not affect the outcome of the algorithm. Note that by this algorithm, the following identities hold for all values of k:

- $A^{(k-1)} = Q^{(k)} R^{(k)}$
- $A^{(k)} = R^{(k)} Q^{(k)}$
- $\left(Q^{(k)}\right)^T Q^{(k)} = I$

The similarity between these two algorithms is illustrated by the following theorem.

Theorem 3.1. The simultaneous iteration method and the QR method both generate the same sequences of matrices $A^{(k)}$, $\overline{Q}^{(k)}$, and $\overline{R}^{(k)}$, which satisfy the following relations:

1. $A^k = \overline{Q}^{(k)}\overline{R}^{(k)}$
2. $A^{(k)} = \left(\overline{Q}^{(k)}\right)^T A\overline{Q}^{(k)}$

Proof. We proceed by induction on k.

When $k = 0$, it is clear that $A^{(0)}$, $\overline{Q}^{(0)}$, and $\overline{R}^{(0)}$ are the same for both the simultaneous iteration method algorithm and the QR method algorithm, and these values satisfy Properties 1 and 2.

Suppose now that the values of these matrices are the same for both algorithms for some iteration $k-1$, and that they satisfy Properties 1 and 2 in this iteration.

In the simultaneous iteration method, we have

$$\begin{aligned} A^k &= AA^{k-1} \\ &= A\left(\overline{Q}^{(k-1)}\overline{R}^{(k-1)}\right) \\ &= \left(\overline{Q}^{(k)}R^{(k)}\right)\overline{R}^{(k-1)} \\ &= \overline{Q}^{(k)}\overline{R}^{(k)} \end{aligned}$$

and thus Property 1 is satisfied on the kth iteration; Property 2 is satisfied directly by definition of the simultaneous iteration algorithm.

In the QR method, we have

$$\begin{aligned} A^k &= AA^{k-1} \\ &= A\left(Q^{(1)}Q^{(2)}Q^{(3)}\dots Q^{(k-1)}R^{(k-1)}\dots R^{(1)}\right) \\ &= \left(Q^{(1)}R^{(1)}\right)\left(Q^{(1)}Q^{(2)}Q^{(3)}\dots Q^{(k-1)}R^{(k-1)}\dots R^{(1)}\right) \\ &= Q^{(1)}\left(Q^{(2)}R^{(2)}\right)Q^{(2)}Q^{(3)}\dots Q^{(k-1)}R^{(k-1)}\dots R^{(1)} \\ &= Q^{(1)}Q^{(2)}\left(Q^{(3)}R^{(3)}\right)Q^{(3)}\dots Q^{(k-1)}R^{(k-1)}\dots R^{(1)} \\ &= \dots \\ &= Q^{(1)}Q^{(2)}Q^{(3)}\dots Q^{(k-1)}\left(Q^{(k)}R^{(k)}\right)R^{(k-1)}\dots R^{(1)} \\ &= \overline{Q}^{(k)}\overline{R}^{(k)}, \end{aligned}$$

which proves that Property 1 holds for the kth iteration of the QR method.

We also have that

$$\begin{aligned} A^{(k)} &= R^{(k)}Q^{(k)} \\ &= \left(\left(Q^{(k)}\right)^T Q^{(k)}\right)R^{(k)}Q^{(k)} \\ &= \left(Q^{(k)}\right)^T A^{(k-1)}Q^{(k)} \\ &= \left(Q^{(k)}\right)^T \left(\left(\overline{Q}^{(k-1)}\right)^T A\overline{Q}^{(k-1)}\right)Q^{(k)} \\ &= \left(\overline{Q}^{(k)}\right)^T A\overline{Q}^{(k)}, \end{aligned}$$

which proves that Property 2 holds for the kth iteration of the QR method as well.

By hypothesis, the values of $A^{(k-1)}$, $\overline{Q}^{(k-1)}$, and $\overline{R}^{(k-1)}$ were the same for both algorithms. Since both algorithms satisfy Property 1 on the kth iteration, we have $A^k = \overline{Q}^{(k)}\overline{R}^{(k)}$ for both algorithms, and since the QR decomposition is unique, it follows that $\overline{Q}^{(k)}$ and $\overline{R}^{(k)}$ are also the same for both algorithms. Both algorithms also satisfy Property 2 on the kth iteration, which means that the matrix $A^{(k)} = (\overline{Q}^{(k)})^T A\overline{Q}^{(k)}$ is also the same for both algorithms on the kth iteration.

Hence, both algorithms produce the same values for the matrices $A^{(k)}$, $\overline{Q}^{(k)}$, and $\overline{R}^{(k)}$ which satisfy Properties 1 and 2. By induction, this holds for all k, proving the theorem. □

Now, since we saw that the columns of $\overline{Q}^{(k)}$ converged to a basis of eigenvectors of A in the simultaneous iteration method, the result of the above theorem tells us that the same holds for the columns of $\overline{Q}^{(k)}$ as computed using the QR algorithm. In particular, the columns $\overline{q}_i$ of $\overline{Q}^{(k)}$ each converge to a unit eigenvector q_i of A.

Property 2 of the theorem tells us that

$$A_{ij}^{(k)} = \left(\overline{q}_i^{(k)}\right)^T A\overline{q}_j^{(k)}$$

where $\overline{q}_i^{(k)}$ and $\overline{q}_j^{(k)}$ are columns i and j, respectively, of $\overline{Q}^{(k)}$. But, we saw that $\overline{q}_i^{(k)} \to q_i$ and $\overline{q}_j^{(k)} \to q_j$ as $k \to \infty$, where q_i and q_j are unit eigenvectors of A and are orthogonal if $i \neq j$.

In the case that $i \neq j$, we have that

$$A_{ij}^{(k)} \to q_i^T Aq_j = \lambda_j q_i^T q_j = 0,$$

since the eigenvectors are orthogonal; thus $A^{(k)}$ approaches a diagonal matrix.

In the case that $i = j$, we have that

$$A_{ii}^{(k)} \to q_i^T Aq_i = \lambda_i q_i^T q_i = \lambda_i,$$

since the eigenvectors are unit vectors; thus the diagonal elements of $A^{(k)}$ approach the eigenvalues of A.

We finally see that the QR iteration method provides an orthogonal matrix $\overline{Q}^{(k)}$ whose columns approach eigenvectors of A, and a diagonal matrix $A^{(k)}$ whose diagonal elements approach the eigenvalues of A. Thus the QR method presents a simple, elegant algorithm for finding the eigenvectors and eigenvalues of a real, symmetric, full rank matrix.

The QR method is, in essence, the same as the simultaneous iteration method, and therefore suffers the same restriction on the matrices it can be applied to. Unfortunately, these two methods may fail when there are non-real eigenvalues, or when the eigenvectors do not form an orthogonal basis of $\mathbb{R}^n$. However, symmetric matrices with real entries do occur frequently in many physical applications, so there is some use for these algorithms.

4 Conclusions and Improvements

The purpose of this paper was to provide an overview of some of the numeric techniques used to compute eigenvalues and eigenvectors of matrices. These methods are all based on simple ideas that were steadily generalized and adapted until they became powerful iterative algorithms.

However, even the QR iteration method as presented in this paper is generally not suitable for use in practice, even in the situations when it can be applied. There are many ways to refine the algorithms we have seen in order to speed up the implementations. For example, before applying the QR algorithm, it is common to reduce the matrix A into a simpler form, such as a tridiagonal matrix: performing QR decompositions on the new matrix then becomes much faster. It is also possible to improve the QR iteration method by incorporating shifts and Rayleigh quotients, just as these concepts helped improve the original

power iteration method. The variations that are implemented in practice are vastly more sophisticated than the simple algorithms presented here.

As with all algorithms, the question of improvement is always an open problem. Modern information processing deals with increasingly large matrices, and efficient methods for computing eigenvalues and eigenvectors are extremely necessary. The techniques of the future may become much more complicated in order to keep up with this growing demand, but as we have seen, there is a lot that can be done using simple ideas and well-chosen generalizations.

5 Acknowledgments

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De Broglie-Bohm and Feynman Path Integrals

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Abstract: The de Broglie-Bohm theory offers what is arguably the clearest and most conceptually coherent formulation of nonrelativistic quantum mechanics known today. It not only renders entirely unnecessary all of the unresolved paradoxes at the heart of orthodox quantum theory, but moreover, it provides the simplest imaginable explanation for its entire (phenomenologically successful) mathematical formalism. All this, with only one modest requirement: the inclusion of precise particle positions as part of a complete quantum mechanical description. In this paper, we propose an alternative proof to a little known result—what we shall refer to as the de Broglie-Bohm path integral. Furthermore, we will show explicitly how the more famous Feynman path integral emerges and is, in fact, best understood as a consequence thereof.

1 Introduction

Despite its enormous success in the prediction of experimental regularities, orthodox quantum theory [Dir58, vN55] has been plagued, ever since its inception, by a wide range of conceptual difficulties. The most famous of these has come to be known as the measurement problem [Bel87]. That is, namely, the question of why a theory of fundamental physics should give axiomatic significance to such human-centered concepts as 'observers,' and why should it not be possible to understand 'measurement' as a particular case of some general evolution process rather than need to allocate a special postulate for it [dM02].

While many still regard the measurement problem as the principal foundational issue at the heart of orthodox quantum theory, it is perhaps best understood merely as a manifestation of a more fundamental conceptual inadequacy: that is, simply put, its inability to provide a clear notion of what quantum mechanics is actually *about*.

The central claim made by orthodox quantum theory is that the wavefunction provides a *complete* description of a quantum system. This would seem to entail that quantum mechanics is, fundamentally, about wavefunctions. However, accepting such a proposition faces us with an inescapable problem, most eloquently formulated by Schrödinger within the context of his famous cat paradox: namely, the question of what it actually means for an object to literally exist in a superposition of eigenstates of a measurement operator [Sch35]. The failure of orthodox quantum theory to offer any sort of coherent resolution to concerns of this sort is largely the reason for which it has continually remained so ambiguous and obscure.

Taking all of this into account, it may come as a surprise that very little is needed in order to essentially do away with all of the above perplexities. The de Broglie-Bohm theory (sometimes referred to as Bohmian mechanics or pilot-wave theory), proposed by Louis de Broglie in 1927 [dB27] and extended by David Bohm in 1952 [Boh52], accomplishes just that, in the simplest imaginable way. If, instead of wavefunctions, quantum mechanics really describes the behaviour of electrons and other elementary particles, then it seems that quantum mechanics should be, fundamentally, about particles in motion. As such, the essential insight of de Broglie-Bohm theory is that, in addition to the wavefunction, the description of a quantum system should also include its configuration—that is to say, the *precise* positions of all the particles at all times.

What naturally results from this is a deterministic quantum theory of particle trajectories—one which, as we shall see, not only accommodates, but provides the simplest known *explanation* for the quantum formalism (including the Born rule, the Heisenberg uncertainty relation, the representation of dynamical variables as Hilbert space operators, and so on).

This paper is organized as follows. Section 2 details the historical development of de Broglie-Bohm theory, and then Section 3 provides an exposition of its basic axioms. Section 4 is devoted to Bohm's second-order formulation of the theory in terms of the so-called quantum potential. In Section 5, we discuss, in broad outline, the notion of quantum equilibrium and the empirical equivalence between de Broglie-Bohm theory and orthodox quantum theory, while Section 6 briefly addresses some of the main criticisms which have been raised against the former. Then, in Section 7, we propose our alternative proof of the de Broglie-Bohm path integral, and in Section 8 we show how this naturally leads to the formulation of the Feynman path integral. Finally, Section 9 concludes the paper.

2 Historical Perspectives

At the 1927 Solvay conference in Brussels—perhaps the most important meeting in the history of quantum theory—Louis de Broglie presented what he called 'the new dynamics of quanta' [BV09]. This, the culmination of his independent work beginning in 1923 [dB23] and supplemented by Schrödinger's discoveries in 1926 [Sch26], amounted to the basic axioms of what is now referred to as de Broglie-Bohm theory.

De Broglie's fundamental insight was the following. In considering diffraction (the apparent bending of light when passing through narrow slits) which experiments had then demonstrated for X-rays, he argued [Val09] that this sort of phenomenon should not be understood—as in the classical picture—to be a wave-like manifestation of light itself. Rather, it should be viewed simply as evidence that photons do not always move in a straight line in empty space. What this amounts to, then, is a failure of Newton's first law, thereby requiring an entirely new form of dynamics—one that is based on *velocity*, not acceleration.

By the time he wrote his 1924 doctoral thesis [dB24], de Broglie had discovered precisely that: a unified dynamical theory of particles and waves, one where the latter guide the former along trajectories via a first-order law of motion (what is now referred to as the guiding equation). Moreover, this 'new dynamics' led him to making the extraordinary prediction [Val09] that material bodies (such as electrons) would also undergo diffraction, just like photons. (This was confirmed a few years later by Clinton Davisson and Lester Germer in their experiments on the scattering of electrons by crystals and, consequently, earned de Broglie the 1929 Nobel Prize for Physics.)

Inspired by this work, it was Schrödinger who, in 1926, developed the celebrated equation (now bearing his name) for de Broglie's waves. (This, despite the fact that Schrödinger dropped the trajectories from the theory, and considered only the waves.) Meanwhile, de Broglie unsuccessfully tried to derive the guiding equation from a still deeper theory, but by 1927, contented himself with presenting it as a provisional axiom of his 'new dynamics' [dB27].

Contrary to common misconceptions, de Broglie's theory was no less extensively discussed at the Solvay conference than any of the alternative proposals, with support expressed by Einstein and Brillouin [BV09]. Nevertheless, de Broglie abandoned his ideas a few years later. A large part of the reason for this—again, contrary to misconceptions—had to do with the fact that he could not construct an analysis of the quantum measurement process from his theory (and thereby show explicitly how it reproduces all of the successful empirical predictions of orthodox quantum theory). That problem was resolved in 1952 by Bohm [Boh52].

What then became known as the de Broglie-Bohm theory was essentially ignored for most of the second half of the last century, despite ceaseless efforts by John Bell to promote it as "an antidote to the prevailing complacency" [Bel87]. However, the situation has changed considerably over the course of the past two decades. The early 1990s saw the publication of the first textbooks to present quantum mechanics from this perspective [BH93, Hol93], and the number of physicists working on it has since increased substantially. The de Broglie-Bohm theory is now accepted as an alternative—if little used—formulation of quantum mechanics.

Now we are ready to state—in full mathematical detail—the theory's axioms, which are typically formulated in more or less the same way as de Broglie presented them in 1927.

3 The Axioms of de Broglie-Bohm Theory

For any system of N nonrelativistic spinless particles, there exists an associated wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ belonging to the set of complex-valued functions which are square-integrable on the configuration space

$$\Omega = \left\{ q = (\mathbf{q}_1, \ldots, \mathbf{q}_N) | \, \mathbf{q}_j = (q_{j1}, q_{j2}, q_{j3}) \in \mathbb{R}^3 \right\} \subseteq \mathbb{R}^{3N}.$$

The de Broglie-Bohm theory postulates [BH93, Hol93, DT09, BDDDZ95] that the system is *completely described* by its wavefunction $\psi(q,t)$ and its configuration $\mathcal{Q}(t) = (\mathbf{Q}_1(t), \ldots, \mathbf{Q}_N(t)) \in \Omega$ (where $\mathbf{Q}_j(t) \in \mathbb{R}^3$ is the position of the j-th particle).

The theory is defined by the following two axioms:

Axiom 1 (The Schrödinger Equation). The wavefunction evolves via

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \nabla_j^2 \psi + V\psi, \tag{3.1}$$

where $\hbar$ is the reduced Planck constant, m_j is the mass of the j-th particle, $V : \Omega \to \mathbb{R}$ is the potential, and $\nabla_j = \partial / \partial \boldsymbol{q}_j$.

Axiom 2 (The Guiding Equation). The position of the j-th particle is determined by the equation of motion

$$\frac{d\mathbf{Q}_j}{dt} = \frac{\hbar}{m_j} \Im \left(\frac{\nabla_j \psi}{\psi} \right) \Big|_{\mathcal{Q}(t)}. \tag{3.2}$$

Equivalently, for any wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ there exists an associated velocity field $v^\psi = (\mathbf{v}_1^\psi, \ldots, \mathbf{v}_N^\psi) : \Omega \to \Omega$ defined according to

$$\mathbf{v}_j^\psi = \frac{\hbar}{m_j} \Im \left(\frac{\nabla_j \psi}{\psi} \right),$$

such that the configuration of the system follows

$$\frac{d\mathcal{Q}}{dt} = v^\psi \big(\mathcal{Q}(t) \big).$$

Equations 3.1 and 3.2 form a complete specification of the theory, giving rise to a clear and astonishingly simple picture of the evolution of a de Broglie-Bohm universe: particles (a concept taken in its most literal sense) move along trajectories (solutions of Equation 3.2) that are choreographed by the wavefunction (the solution of Equation 3.1).

We stress that there is no need—and, indeed, no room—for any further axioms. In particular, no 'measurement' axioms (a concept central to orthodox quantum theory) are necessary. In de Broglie-Bohm theory, there is no special role for 'the observer,' nothing to elevate the status of a 'measurement' above that of any other physical process.

Furthermore, de Broglie-Bohm theory is deterministic. While the issue of determinism was never a primary motivation for its creation [BH93], it so happens that the simplest possible quantum theory of particle motion has this feature.

Now, before addressing the non-trivial question of how such a theory could possibly reproduce all of the well-known probabilistic results of orthodox quantum theory (that will be left for Section 5), we consider an even more immediate concern: supposing that we indeed accept trajectories as indispensable to a complete quantum mechanical description, what reason is there for the particular choice of Equation 3.2 as the basic law of motion? As we have seen, de Broglie eventually recognized no alternative to simply giving the guiding equation the same status as the Schrödinger equation—namely, that of an axiom with

no more ultimate justification—and yet, remarkably, there are a number of different lines of reasoning by which the former can be shown to naturally emerge from the latter.

We now present one of the more common approaches to resolving this issue. Consider again a system of N particles with some wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$. With a suitable choice of a phase function $S : \Omega \times \mathbb{R} \to \mathbb{R}$, we can rewrite the wavefunction in the polar form

$$\psi = |\psi| \exp\left(\frac{i}{\hbar} S\right). \tag{3.3}$$

Inserting Equation 3.3 into the Schrödinger equation, we obtain, after separating real and imaginary parts,

$$\frac{\partial |\psi|^2}{\partial t} + \sum_{j=1}^{N} \nabla_j \cdot \left(|\psi|^2 \frac{\nabla_j S}{m_j}\right) = 0, \tag{3.4}$$

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} + V + \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \left(\frac{\nabla_j^2 |\psi|}{|\psi|}\right) = 0. \tag{3.5}$$

While Equation 3.4 is just the familiar continuity equation (expressing conservation of probability), we notice that Equation 3.5 is reminiscent of the following result from classical mechanics:

Theorem 3.1 (The Jacobi Theorem). In a system of N particles obeying classical mechanics, if $S : \Omega \times \mathbb{R} \to \mathbb{R}$ is any complete solution of the partial differential equation

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} + V = 0, \tag{3.6}$$

known as the Hamilton-Jacobi equation, then the equation of motion

$$\frac{d\mathbf{Q}_j}{dt} = \left.\frac{\nabla_j S}{m_j}\right|_{\mathcal{Q}(t)} \tag{3.7}$$

can solved to yield the system's configuration $\mathcal{Q}(t) = (\mathbf{Q}_1(t), \ldots, \mathbf{Q}_N(t)) \in \Omega$, where $\mathbf{Q}_j(t) \in \mathbb{R}^3$ is the position of the j-th particle.

Proof. See Takhtajan's book [Tak08]. □

Now, observe that Equation 3.6 is identical to Equation 3.5 except for the fourth term on the left hand side of the latter. We will address the precise meaning of this quantity in the next section. For now we can immediately deduce, as it is on the order of the square of the Planck constant, that it will be negligible at macroscopic distances (i.e. where classical mechanics is presumed to apply). That is to say, we can interpret Equation 3.6 as the classical limit of Equation 3.5, which is consequently referred to as the *quantum* Hamilton-Jacobi equation. But if that is the case, then the function S in Equation 3.7 must be the same as the phase function from Equation 3.3. Combining these two relations gives us, after a bit of algebra, Equation 3.2 from Axiom 2. (Another approach for obtaining this result, it turns out, is just to seek the simplest first-order equation of motion dependent on the wavefunction that preserves Galilean and time-reversal invariance [DGZ92].)

Remark 3.2: The above considerations give us many different ways of writing the components of the velocity field v^ψ,

$$\mathbf{v}_j^\psi = \frac{\hbar}{m_j} \Im\left(\frac{\nabla_j \psi}{\psi}\right) = \frac{\nabla_j S}{m_j} = \frac{\boldsymbol{\mathcal{J}}_j^\psi}{|\psi|^2}, \tag{3.8}$$

where the reader familiar with orthodox quantum theory will recognize the the quantum probability current $\mathcal{J}^\psi = (\boldsymbol{\mathcal{J}}_1^\psi, \boldsymbol{\mathcal{J}}_2^\psi, \ldots, \boldsymbol{\mathcal{J}}_N^\psi) : \Omega \to \Omega$ defined according to $\boldsymbol{\mathcal{J}}_j^\psi = (i\hbar/2m_j)\left(\psi \nabla_j \psi^* - \psi^* \nabla_j \psi\right)$.

4 The Quantum Potential and the Bohm Equation of Motion

It is sometimes useful to reformulate de Broglie-Bohm theory in terms of the following:

Definition 4.1 (The Quantum Potential). For an N-particle system with wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$, the quantum potential is defined to be

$$V_q = \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \left(\frac{\nabla_j^2 |\psi|}{|\psi|} \right). \tag{4.1}$$

With this definition, we may suggestively write:

Theorem 4.1 (The Bohm Equation of Motion). The trajectory of the j-th particle is determined by

$$m_j \frac{d^2 \mathbf{Q}_j}{dt^2} = -\nabla_j \big(V + V_q\big)\big|_{\mathcal{Q}(t)}. \tag{4.2}$$

Proof. The above follows directly from taking the time derivative of both sides of Equation 3.2. □

We note here that, in contrast to de Broglie, Bohm regarded Equation 4.2 as the theory's fundamental equation of motion, with Equation 3.2 taken simply as a constraint [BH93]. The justification, often seen as somewhat contrived, proceeds along the following lines: since Equation 3.6 is interpreted as being the classical limit of Equation 3.5, the meaning of the term V_q in the latter is necessarily that of an additional potential to the classical potential V; from this, the quasi-Newtonian law of motion in Equation 4.2 logically follows.

However, it is rather unnatural to recast de Broglie-Bohm theory in this way, i.e. as a second-order theory. Indeed, as we have seen, de Broglie's essential motivation for its formulation had been the need for a new (non-Newtonian) *velocity-based* dynamics, the essence of which is captured by the guiding equation. Nevertheless, the Bohm equation of motion is a useful result insofar as it immediately allows us to draw the following observation:

Remark 4.2: In the macroscopic limit, $V_q \to 0$ and so the Bohm equation of motion reduces to Newton's second law i.e. de Broglie-Bohm theory reduces to classical mechanics.

Furthermore, we will later on need to make use of the so-called Lagrangian function. This is a notion that originates from classical mechanics, where it is defined as follows:

Definition 4.2 (The Classical Lagrangian). The Lagrangian function in classical mechanics is defined as

$$\mathcal{L}_c = \sum_{j=1}^{N} \frac{1}{2} m_j \left\| \mathbf{v}_j^{\psi} \right\|^2 - V, \tag{4.3}$$

where $\mathbf{v}_j^{\psi}$ is the velocity of the j-th particle.

In fact, it can be shown that knowing the Lagrangian function is sufficient for obtaining a complete description of any system in classical mechanics (insofar as all of the equations of motion for that system can be derived directly therefrom). Therefore, it is possible and often quite convenient to formulate all of classical mechanics in terms of this quantity.

Now, by analogy, we can also define:

Definition 4.3 (The Quantum Lagrangian). Inspired by the classical Lagrangian function, we define the quantum Lagrangian to be

$$\mathcal{L}_q = \sum_{j=1}^{N} \frac{1}{2} m_j \left\| \mathbf{v}_j^{\psi} \right\|^2 - (V + V_q), \tag{4.4}$$

with $\mathbf{v}_j^{\psi}$ and V_q given by Equations 3.8 and 4.1 respectively.

Just as in the classical case, it is certainly possible to formulate de Broglie-Bohm theory in terms of Equation 4.4. However, we do not wish to enter into the details of that here; the usefulness of the above two definitions will, for our purposes, become apparent in Sections 7 and 8.

For the moment, however, we turn our attention to the question of how the notions of randomness and uncertainty enter into de Broglie-Bohm theory.

5 Quantum Equilibrium and the Emergence of Absolute Uncertainty

As noted before, de Broglie-Bohm theory describes a universe which is completely deterministic. Yet, it is well known that in our universe, all quantum phenomena appear to systematically yield random outcomes. Orthodox quantum theory accounts for this as follows. A quantum system, therein presumed to be completely described by its wavefunction $\psi(q,t)$, evolves (deterministically) according to the Schrödinger equation, *until* such a time as a 'measurement' is performed. Then, the wavefunction randomly 'collapses' onto one of all the possible outcomes, the statistical distribution function for which is given by $\rho(q,t) = |\psi(q,t)|^2$. This is the so-called Born rule.

Now, while Section 1 has already touched upon the difficulties that arise from ascribing axiomatic status to the Born rule, it does nevertheless—so far as we know—prescribe the correct probabilities for the outcomes of quantum measurements. Put differently, our world indeed looks to us random, with probability distribution $\rho = |\psi|^2$. However—and we stress this yet again—it is a randomness that manifests itself *at no times other than* during 'measurements'. Therefore, rather than draw from this the (all too common) conclusion that there is some sort of fundamental element of chance to how our universe works, it seems far more natural to interpret the notion of probability (associated with the outcomes of measurements) as an emergent feature of some underlying dynamics—a description of which is precisely what de Broglie-Bohm theory has to offer.

The axiomatic status of the Born rule in orthodox quantum theory seems even more bewildering when we consider that this concept of emergent probability is nothing new in physics. Indeed, the same notion lies at the basis of classical statistical mechanics, where the underlying (deterministic) Newtonian dynamics gives rise to the apparently random behaviour of large systems (such as, for instance, the famous Maxwell-Boltzmann distribution of molecular velocities in a gas).

So, if we expect quantum probabilities to likewise have a dynamical origin and the correct account thereof to be that which is given by de Broglie-Bohm theory, then the question which remains to be addressed is this: how exactly does the $\rho = |\psi|^2$ distribution arise therefrom? It turns out that the solution to this problem is very non-trivial; its satisfactory understanding requires exhaustive analysis [DGZ92, Val01, VW05], to which we can provide here only an outline.

The discussion hinges upon the notion of equilibrium as a general feature of dynamical systems. Broadly speaking, a system is said to be in a statistical state of equilibrium (for example, thermodynamic equilibrium in classical statistical mechanics) once it has reached some measure of stationarity—that is to say, a time-invariant distribution. It turns out that in de Broglie-Bohm theory, $\rho = |\psi|^2$ fits precisely this description, and in this context is referred to as being *equivariant*. Formally, this can be stated as:

Theorem 5.1 (The Equivariance Theorem). If $\rho(q,t_0) = |\psi(q,t_0)|^2$ for some initial time t_0, then $\rho(q,t) = |\psi(q,t)|^2$, $\forall t$.

Proof. This follows trivially from Equation 3.4. □

However, even more can be said:

Theorem 5.2 (The Uniqueness of Quantum Equilibrium). $\rho = |\psi|^2$ is the unique equivariant distribution that is also a local functional of ψ.

Proof. See Goldstein and Struyve's paper [GS07]. □

The above theorems suggest that if indeed equivariance is the required condition for quantum equilibrium—that is, an equilibrium state of the system relative to the wavefunction—then its only possible measure is the $\rho = |\psi|^2$ distribution, thus leading to:

Claim 5.1 (The Quantum Equilibrium Hypothesis). For a system in quantum equilibrium having wavefunction ψ, the statistical distribution of its coordinates is given by

$$\rho(q,t) = |\psi(q,t)|^2 . \tag{5.1}$$

This result—whose full justification [DGZ92, Val01, VW05] is, once again, beyond the scope of this paper—gives rise to the notion of absolute uncertainty (which is essentially equivalent to the Born rule): that it is impossible to know more about the configuration of a system beyond that which Equation 5.1 allows. In other words, a universe governed by (deterministic) de Broglie-Bohm dynamics evolves so that, when in quantum equilibrium, the *appearance* of randomness emerges (in much the same way as it does, when in thermodynamic equilibrium, for one governed by classical statistical mechanics).

Now, the natural question that arises from the above argument is this: can there exist systems that are *not* in quantum equilibrium, i.e. for which $\rho \neq |\psi|^2$? While orthodox quantum theory deems this inadmissible (it being an explicit violation of the Born rule), there is nothing in de Broglie-Bohm theory preventing such a possibility. However, numerical simulations [VW05] have demonstrated the effective relaxation of such systems to the equilibrium distribution $\rho = |\psi|^2$ (where their discrepancy therewith was found to decrease approximately exponentially over time).

Moreover, it has recently been proposed [Val07, Val10] that our universe began with a non-equilibrium distribution $\rho \neq |\psi|^2$ (as there is, a priori, no reason to favour an initial equilibrium state), with sufficient time to then relax to the $\rho = |\psi|^2$ distribution we observe today. This avenue of exploration is still in its early stages, but (if the above claim turns out to be true) may soon offer the first possibility of an empirical distinction between de Broglie-Bohm theory and orthodox quantum theory via observations of the early universe.

Otherwise, in our present universe (i.e. which is presumed to have long reached quantum equilibrium), Claim 5.1 is sufficient to guarantee the emergence of the usual mathematical formalism of quantum mechanics—operators as observables and all the rest—from de Broglie-Bohm theory. (The reader is referred to Dürr, et al. [DGZ04] for an analysis of how this comes about in detail.) Therefore, de Broglie-Bohm theory not only reproduces all of the familiar predictions of orthodox quantum theory, but it does so in such a manner as to eliminate the measurement problem (see Goldstein, et al. [GTZ10] for a simple treatment of this) as well as any dubious notions regarding what the reality of a quantum system might actually be.

6 Criticisms of de Broglie-Bohm Theory

We turn briefly to a consideration of the main criticisms presently facing de Broglie-Bohm theory. Perhaps the most serious among these is that there are well-known issues surrounding the attempts to find for it a Lorentz invariant extension. What makes this problematic is the nonlocality of the theory manifested through the guiding equation, i.e. the fact that the trajectory of each particle depends *instantaneously* on the positions of *all* the particles in the system. (In fact, Bell showed [Bel64] that nonlocality is a necessary feature of quantum mechanics under *any* interpretation, and considered it "a merit of the de Broglie-Bohm version to bring this out so explicitly that it cannot be ignored" [Bel87].) Despite the obvious problems that this imposes, a number of limited approaches towards a relativistic generalization of the theory have so far been proposed [BH93, BDGZ96, DGMBZ99], making a complete rejection of de Broglie-Bohm theory on this basis rather premature. (There are, on the other hand, interesting arguments to the effect that, due to the first-order structure of the theory, it is an interpretational mistake to even conceive of Lorentz invariance as a fundamental symmetry; the search for it, then, would be essentially misguided [Val97].) Furthermore, contrary to common suspicions, an extension of de Broglie-Bohm theory to quantum field theories does turn out to be possible as well [Bel87, DGTZ04].

Most of the other criticisms which have been voiced against this theory (for example, that it is 'complicated' or 'contrived,' that it constitutes a regression back to classical physics, that it does not postulate the quantum equilibrium hypothesis, and other such assertions) lack much of a compelling basis and will not be discussed any further here. (For detailed responses to objections of this sort, the reader is referred to articles by Passon and Kiessling [Pas05, Kie10].)

7 The de Broglie-Bohm Path Integral

A fundamental problem in quantum mechanics is the propagation of the wavefunction: that is, given the wavefunction $\psi(q,t)$ at some initial time t, we desire to calculate $\psi(q',t')$, $\forall q' \in \Omega$ at any future time t', assuming its evolution to be governed by the Schrödinger equation, i.e. Equation 3.1. It turns out that in certain simple situations (such as when $V=0$), this is something that can be resolved without much effort [Tak08]. However, for an arbitrary potential V, the problem becomes non-trivial and so, in order to formulate a solution, we shall have to construct so-called quantum mechanical path integrals. That is the aim of this and the next section; in particular, we will first consider the solution from the perspective of de Broglie-Bohm theory, and then we will show how to obtain from it the usual solution from orthodox quantum theory.

The propagation of the wavefunction in the context of de Broglie-Bohm theory has recently been considered and achieved via what we shall call the de Broglie-Bohm path integral [AG03]. Our aim here is to provide for it an alternative proof.

Consider again the wavefunction in polar form, i.e. Equation 3.3. In order to propagate it in this representation, we shall propagate separately its modulus as well as its phase function. Between any $q \in \Omega, t \in \mathbb{R}$ and any $q' \in \Omega, t' \in \mathbb{R}$ along de Broglie-Bohm trajectories, we obtain the following:

Lemma 7.1. The wavefunction modulus is propagated according to

$$|\psi(q',t')| = \left[\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\int_{t}^{t'} \nabla\cdot\mathbf{v}_j^{\psi}\,dt\right)\right]|\psi(q,t)|, \tag{7.1}$$

where $\mathbf{v}_j^{\psi}$ is given by Equation 3.8.

Proof. The continuity equation, i.e. Equation 3.4, can be rewritten as

$$\frac{\partial|\psi|}{\partial t} = -\frac{|\psi|}{2}\sum_{j=1}^{N}\nabla_j\cdot\mathbf{v}_j^{\psi}.$$

Now, separating and integrating along our de Broglie-Bohm paths, we get

$$\ln\left|\frac{\psi(q',t')}{\psi(q,t)}\right| = -\frac{1}{2}\sum_{j=1}^{N}\int_{t}^{t'}\nabla\cdot\mathbf{v}_j^{\psi}\,dt,$$

from which Equation 7.1 immediately follows. □

Lemma 7.2. The phase function is propagated according to

$$S(q',t') = S(q,t) + \int_{t}^{t'}\mathcal{L}_q\,dt, \tag{7.2}$$

where the quantum Lagrangian $\mathcal{L}_q$ is given by Equation 4.4.

Proof. Consider the time rate of change of the function S (i.e. the total derivative dS/dt) along the trajectories. By the chain rule, we know that this can be decomposed into contributions from the motion of all the particles (along their respective paths) plus its time rate of change at a fixed configuration (i.e. the partial derivative $\partial S/\partial t$); that is to say,

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\partial S}{\partial \boldsymbol{q}_j} \cdot \frac{d\boldsymbol{q}_j}{dt} + \frac{\partial S}{\partial t}.$$

As we are evaluating dS/dt along de Broglie-Bohm paths, the terms $d\boldsymbol{q}_j/dt$ are simply the components of the velocity field given by Equation 3.8. This means that we can write $d\boldsymbol{q}_j/dt = \nabla_j S/m_j$, and therefore,

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{m_j} + \frac{\partial S}{\partial t}.$$

Now, using the expression for $\partial S/\partial t$ given by the quantum Hamilton-Jacobi equation, i.e. Equation 3.5, we have

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} - (V + V_q) = \mathcal{L}_q.$$

In other words, the total time derivative of S along de Broglie-Bohm paths is simply the quantum Lagrangian. Finally, we can substitute this into

$$S(q', t') = S(q, t) + \int_t^{t'} \left(\frac{dS}{dt}\right) dt$$

to obtain precisely Equation 7.2. □

Now we put together Lemmas 7.1 and 7.2, thereby allowing us to finally prove the main result of this paper:

Theorem 7.1 (The de Broglie-Bohm Path Integral). The wavefunction is propagated via the de Broglie-Bohm path integral according to

$$\psi(q', t') = \exp\left[\int_t^{t'} \left(\frac{i}{\hbar}\mathcal{L}_q - \frac{1}{2}\sum_{j=1}^{N} \nabla_j \cdot \mathbf{v}_j^{\psi}\right) dt\right] \psi(q, t).$$

Proof. By Lemma 7.2, we have that

$$\exp\left[\frac{i}{\hbar} S(q', t')\right] = \exp\left(\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt\right) \exp\left[\frac{i}{\hbar} S(q, t)\right].$$

Combining the above with Lemma 7.1, we get

$$|\psi(q', t')| \exp\left[\frac{i}{\hbar} S(q', t')\right] = \exp\left(\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{N} \int_t^{t'} \nabla \cdot \mathbf{v}_j^{\psi} dt\right) |\psi(q, t)| \exp\left[\frac{i}{\hbar} S(q, t)\right].$$

Recalling the polar form from Equation 3.3, this simply means that

$$\psi(q', t') = \exp\left[\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt - \frac{1}{2}\sum_{j=1}^{N} \int_t^{t'} \nabla \cdot \mathbf{v}_j^{\psi} dt\right] \psi(q, t),$$

as desired. □

8 The Feynman Path Integral from Theorem 7.1

We have thus far seen how it is possible to construct a path integral formalism for the de Broglie-Bohm theory—one which has, at its heart, the notion of precise particle paths. As mentioned before, orthodox quantum theory provides its own solution for propagating the wavefunction—namely, the Feynman path integral [Fey48] (to be stated explicitly below) which, moreover, also turns out to have useful applicability in quantum field theory (for instance, as a method of quantizing the electromagnetic field [Fey49, Fey50]).

Now, despite the fact that de Broglie-Bohm path integrals and Feynman path integrals originate from strikingly different conceptual bases, it still seems natural to pose the question: what exactly does one have to do (if anything at all) with the other? The answer turns out to be quite remarkable: the latter can be constructed directly from an analysis of the former.

Although a "heuristic argument" for this result has already been put forth in [AG03], our aim here is to prove it a bit more carefully and rigorously. For the sake of simplicity, we shall do this for the single particle case; Theorem 7.1 then reduces to

$$\psi\left(\mathbf{q}',t'\right)=\exp\left[\int_{t}^{t'}\left(\frac{i}{\hbar}\mathcal{L}_{q}-\frac{1}{2}\nabla\cdot\mathbf{v}^{\psi}\right)dt\right]\psi\left(\mathbf{q},t\right), \tag{8.1}$$

where the integration is performed over the de Broglie-Bohm path taken by the particle from $(\mathbf{q},t)$ to $(\mathbf{q}',t')$.

We will soon see that the propagation of the wavefunction as prescribed by the Feynman path integral is naturally expressed in terms of a function known as the propagator. So, before proceeding further, we give its definition:

Definition 8.1 (The Propagator). The propagator $K\left(\mathbf{q}',t';\mathbf{q},t\right)$ of the wavefunction ψ is defined such that

$$\psi\left(\mathbf{q}',t'\right)=\int_{\mathbb{R}^{3}}K\left(\mathbf{q}',t';\mathbf{q},t\right)\psi\left(\mathbf{q},t\right)d^{3}\mathbf{q}.$$

Remark 8.1: In orthodox quantum theory, the standard meaning assigned to the propagator (or 'complex probability amplitude') is the following: the function $\left|K\left(\mathbf{q}',t';\mathbf{q},t\right)\right|^{2}$ is the conditional probability distribution of finding the particle at point $\mathbf{q}'$ at time t' provided that it was at point $\mathbf{q}$ at time t [Tak08].

Furthermore, we shall need to make use of the following result:

Theorem 8.2 (The Free Particle Wavefunction). For the free particle, i.e. when $V=0$, the solution to the Schrödinger equation is

$$\psi\left(\mathbf{q},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int_{\mathbb{R}^{3}}\exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{q}\right)\widehat{\psi}\left(\mathbf{p},t\right)d^{3}\mathbf{p}, \tag{8.2}$$

where $\widehat{\psi}=\mathcal{F}\left(\psi\right)$ is the Fourier transform of the wavefunction.

Proof. See Berndl, et al. [BDGZ96]. □

Now we are ready to construct the Feynman path integral:

Theorem 8.3 (The Feynman Path Integral). Let $[t,t']\subset\mathbb{R}$ be divided into n infinitesimal slices $\left\{\left[t_{k},t_{k+1}\right]\middle|\, t_{k+1}-t_{k}=\Delta t,\forall k\right\}_{k=0}^{n-1}$ with $t_{0}=t$ and $t_{n}=t'$; furthermore, let each t_{k} correspond to position coordinate $\mathbf{q}_{k}\in\mathbb{R}^{3}$ with $\mathbf{q}_{0}=\mathbf{q}$ and $\mathbf{q}_{n}=\mathbf{q}'$. Then, the propagator for $\psi\left(\mathbf{q},t\right)$ is

$$K\left(\mathbf{q}',t';\mathbf{q},t\right)=\lim_{n\to\infty}\int\cdots\int_{\mathbb{R}^{3(n-1)}}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{3}{2}n}\exp\left(\frac{i}{\hbar}\int_{t}^{t'}\mathcal{L}_{c}dt\right)\prod_{k=1}^{n-1}d^{3}\mathbf{q}_{k}, \tag{8.3}$$

where the classical Lagrangian $\mathcal{L}_{c}$ is given by Equation 4.3.

Proof. Consider the free particle solution from Theorem 8.2 on $[t_k, t_{k+1}], \forall k$. Observe that, since the time interval is taken to be infinitesimal, this solution is still valid even if the potential V is arbitrary (i.e. not necessarily zero). For this wavefunction, then, the guiding equation can be shown to yield a constant $m\mathbf{v}^\psi = \mathbf{p}$ for the particle's momentum. This immediately implies that $\nabla \cdot \mathbf{v}^\psi = 0$; furthermore, since $V_q = 0$ as well, we will have $\mathcal{L}_q = \mathcal{L}_c$ (i.e. the quantum Lagrangian is the same as the classical one). Hence,

$$\int_{t_k}^{t_{k+1}} \mathcal{L}_q dt = \int_{t_k}^{t_{k+1}} \left(\frac{\|\mathbf{p}\|^2}{2m} - V \right) dt = \left(\frac{m}{2} \left\| \frac{\mathbf{q}_{k+1} - \mathbf{q}_k}{\Delta t} \right\|^2 - V \right) \Delta t,$$

and so Equation 8.1 yields

$$\psi\left(\mathbf{q}_{k+1}, t_{k+1}\right) = \left\{ \exp\left[\frac{i}{\hbar} \left(\frac{m}{2} \frac{\|\mathbf{q}_{k+1} - \mathbf{q}_k\|^2}{\Delta t} - V \Delta t \right) \right] \right\} \psi\left(\mathbf{q}_k, t_k\right).$$

Now, it can be shown (see [GTZ10, BDGZ96]) that the propagator for a wavefunction in the above form (with $\psi\left(\mathbf{q}_k, t_k\right)$ given by Equation 8.2) is

$$K\left(\mathbf{q}_{k+1}, t_{k+1}; \mathbf{q}_k, t_k\right) = \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar} \left(\frac{m}{2} \frac{\|\mathbf{q}_{k+1} - \mathbf{q}_k\|^2}{\Delta t} - V \Delta t \right) \right]. \tag{8.4}$$

From here, we proceed as in all standard proofs of this theorem. The total propagator $K\left(\mathbf{q}', t'; \mathbf{q}, t\right)$ can be obtained via the integral representation

$$K\left(\mathbf{q}', t'; \mathbf{q}, t\right) = \int \cdots \int_{\mathbb{R}^{3(n-1)}} \prod_{k=0}^{n-1} K\left(\mathbf{q}_{k+1}, t_{k+1}; \mathbf{q}_k, t_k\right) \prod_{k=1}^{n-1} d^3\mathbf{q}_k.$$

Substituting Equation 8.4 into this gives

$$K\left(\mathbf{q}', t'; \mathbf{q}, t\right) = \int \cdots \int_{\mathbb{R}^{3(n-1)}} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{3}{2}n} \exp\left[\sum_{k=0}^{n-1} \frac{i}{\hbar} \left(\frac{m}{2} \left\| \frac{\mathbf{q}_{k+1} - \mathbf{q}_k}{\Delta t} \right\|^2 - V \right) \Delta t \right] \prod_{k=1}^{n-1} d^3\mathbf{q}_k.$$

Taking the $n \to \infty$ limit, the summation in the exponential becomes an integral and we recover exactly Equation 8.3, as desired. □

9 Conclusion

The Feynman path integral is conventionally understood as a sum over all (infinite) possible paths connecting $(\mathbf{q}, t)$ and $(\mathbf{q}', t')$, each of these contributing with an amplitude found by integrating the classical Lagrangian. However, these paths are understood not be real paths, i.e. along which the particle actually moves. (Of course, in orthodox quantum theory, such a concept does not even exist). Rather, they are seen merely as mathematical tools useful for computing the evolution of the wavefunction. In this sense, the Feynman path integral is nothing more than a reformulation of the Schrödinger equation.

However, de Broglie-Bohm theory requires a bit more than this to make the quantum picture complete: namely, that the particle actually does move along one of the possible paths, in accordance with the guiding equation. As we have seen (via Equation 8.1), the evolution of the wavefunction can in this case be calculated, quite elegantly, by integrating the quantum Lagrangian along this one single path i.e. the particle's de Broglie-Bohm trajectory. It should then come as no surprise that the Feynman method of summing over all paths can be constructed with the de Broglie-Bohm theory at its basis.

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The Wave Equation and Multi-Dimensional Time

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Abstract: The classical wave equation initial value problem in single and multiple time dimensions is posed and subsequently, the physical and mathematical basis of it is discussed. The Theorem of Asgeirsson is proved and applied to study the wave equation with multiple time dimensions. Further, with the assembly of work by Courant and Hilbert, the well-posedness of such problems is determined in detail.

1 Introduction

Physical theories or interpretations with multiple time dimensions are usually ignored or shunned by most researchers. The standard consensus among physicists has been that such problems are unstable, or hopelessly unpredictive [Wei]. This interpretation has also transferred into the mathematical community, and so, such problems are generally not considered.

Although it has been shown that the canonical 'initial value problem' for the wave equation with more than one time dimension is ill-posed in the sense of Hadamard [RR62, Wei], the recent work of S. Weinstein and W. Craig has presented a sufficient constraint on such problems endowing them with a well-posedness condition near-that of standard single time dimension problems [CW09]. This latter result will be briefly outlined in the conclusion and hopefully will be the subject of a later publication.

Both of the above arguments for the well-posedness of the single and multiple time wave equation are fleshed out in the proceeding sections, with specific care taken so that, with only limited knowledge of partial differential equations, they can be understood by a standard undergraduate audience.

2 The Wave Equation and The Theorem of Asgeirsson

Definition 2.1. In this paper, an *Initial Value Problem (IVP)* is a pair $[A, B]$ where A is a partial differential equation and B is a set of *initial conditions* which must be satisfied by a particular solution of A. Here, an *initial condition* is a set of function values imposed upon the solution at a specified point in its domain.

Definition 2.2. The *Laplacian* with respect to the vector $x = (x_1, x_2, \ldots, x_n)$, is the differential operator

$$\triangle_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to have *compact support* if $f = 0$ on all but a compact subset of $\mathbb{R}^n$. We will denote C^r functions of this variety by C_0^r.

Definition 2.4. A problem is said to be *well-posed (in the sense of Hadamard)* if each of the following hold:

- there exists a solution,
- the solution is unique,
- the solution depends continuously (in the chosen norm) on the data.

If any of the above conditions fail to hold, the problem is said to be *ill-posed*.

2.1 The wave equation in one time dimension

A useful and well-studied equation in modern mathematics, the 'wave equation' applied to a function $u = u(x, y, z, t)$, is defined as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

where we consider $x,\ y,\ z$ to be independent 'space variables' and t to be an independent 'time variable'. This equation is commonly used to describe the propagation of sound, light, and water waves as they move through their respective 3-dimensional medium.

When generalized to $(n+1)$-dimensional euclidean space, the wave equation is commonly seen in the form:

$$\frac{\partial^2 u}{\partial t^2} = u_{tt} = \triangle_x u, \tag{2.1}$$

where $n \geq 1$.

An interesting and relevant fact about Equation 2.1, tying into its physical significance, is that for every $n \geq 1$, the canonical IVP involving it is well posed on a salient class of physically valuable initial conditions. The usual case of this problem is given by the IVP $[(2.1), (2.2)]$.

Initial conditions: Take $f, g : \mathbb{R}^n \to \mathbb{R}$ where $n = 2k$ or $n = 2k+1$, and define

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{2.2}$$

The IVP $[(2.1, (2.2)]$ is commonly called a 'Cauchy Problem' for the wave equation and is well-posed provided that $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$ [RR62].

2.2 The wave equation in many time dimensions

Taking $t = (t_1, t_2, \ldots, t_m)$ and $x = (x_1, x_2, \ldots, x_n)$ the wave equation in many time dimensions is defined as

$$\triangle_t u = \triangle_x u.$$

Equivalently, as we will see useful later, when defining $y = (t_1, \ldots, t_{m-1})$ and redefining $t = t_m$, the above equation may be written

$$\frac{\partial^2 u}{\partial t^2} = (\triangle_x - \triangle_y)\, u. \tag{2.3}$$

As in section 2.1, we analogously construct corresponding initial conditions for Equation 2.3:

Initial conditions: Take $f, g : \mathbb{R}^n \times \mathbb{R}^{m-1} \to \mathbb{R}$ and define

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y). \tag{2.4}$$

We will soon see that $[(2.3), (2.4)]$ with $f \in C_0^k$ and $g \in C_0^k$ is ill-posed for every $k \geq 1$ (unless we have further restrictions upon f and g [CW09]). In order to prove this, we must first present an important theorem.

2.3 The theorem of Asgeirsson

Theorem 2.1 (Asgeirsson). For a solution $u \in C^2$ of the differential equation

$$\frac{\partial^2 u}{\partial t^2} = (\triangle_x - \triangle_y)\, u \quad (\text{where } m = n),$$

the average of $u(x, t_0)$ on a sphere of radius R and center x_0 in x-space is the same as the average of $u(x_0, t)$ on a sphere of radius R and center t_0 in t-space [Fri55].

Proof. For all $\alpha, \beta > 0$, define the ellipsoidal mean of u in $\mathbb{R}^n$

$$I(\alpha, \beta) = \frac{1}{2n\omega_{2n}} \int_{F(\alpha,\beta,x,t)=1} u(x,t)d\omega.$$

Where $F(\alpha, \beta, x, t) = \frac{|x|^2}{\alpha} + \frac{|t|^2}{\beta}$ and ω_k is the volume of the unit sphere in $\mathbb{R}^k$.

Now observe that because $d\omega$ is invariant under affine transformations, through a change of variables $x = \sqrt{\alpha}\eta$ and $t = \sqrt{\beta}\zeta$, we arrive with

$$I(\alpha, \beta) = \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2=1} u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)d\omega.$$

So

$$\begin{aligned}\frac{\partial}{\partial\alpha} I(\alpha, \beta) &= \frac{1}{2n\omega_{2n}\sqrt{\alpha}} \int\limits_{|\eta|^2+|\zeta|^2=1} \sum_{i=1}^{n} u_{x_i}(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)\eta_i d\omega \\ &= \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2=1} \nabla_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta) \cdot \eta d\omega.\end{aligned}$$

Considering the vector $(\nabla_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta), 0_t)$, we get by the divergence theorem

$$\begin{aligned}\frac{\partial}{\partial\alpha} I(\alpha, \beta) &= \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2<1} \triangle_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)d\eta d\zeta \\ &= \frac{1}{2n\omega_{2n}} (\alpha\beta)^{-\frac{n}{2}} \int\limits_{F(\alpha,\beta,y)<1} \triangle_x u(x,t)dxdt. \qquad (2.5)\end{aligned}$$

Similarly, we arrive with

$$\frac{\partial}{\partial\beta} I(\alpha, \beta) = \frac{1}{2n\omega_{2n}} (\alpha\beta)^{-\frac{n}{2}} \int\limits_{F(\alpha,\beta,y)<1} \triangle_t u(x,t)dxdt. \qquad (2.6)$$

So, we observe by Equations 2.5 and 2.6 that $(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta})I(\alpha, \beta) = 0$ since $(\triangle_x - \triangle_t)u = 0$. Thus, we have that $I(\alpha, \beta) = \phi(\alpha + \beta)$ for some $\phi : \mathbb{R} \to \mathbb{R}$, and in particular,

$$I(\alpha, \beta) = I(\beta, \alpha) \quad \text{for all } \alpha, \beta > 0. \qquad (2.7)$$

Furthermore,

$$\begin{aligned}
\lim_{\beta\to 0^+} 2n\omega_{2n} I(\alpha,\beta) &= \int\limits_{|\eta|^2+|\zeta|^2=1} u(\sqrt{\alpha}\eta,0)d\omega \\
&= \left[\frac{d}{dr}\int\limits_{|\eta|^2+|\zeta|^2\le r^2} u(\sqrt{\alpha}\eta,0)d\eta d\zeta\right]_{r=1} \\
&= \left[\frac{d}{dr}\int\limits_{|\eta|^2\le r^2} u(\sqrt{\alpha}\eta,0)\int\limits_{|\zeta|^2\le r^2-|\eta|^2} d\zeta d\eta\right]_{r=1} \\
&= \left[\frac{d}{dr}\omega_n\int\limits_{|\eta|^2\le r^2} u(\sqrt{\alpha}\eta,0)(r^2-|\eta|^2)^{\frac{n}{2}}d\eta\right]_{r=1} \\
&= n\omega_n\int\limits_{|\eta|^2\le 1} u(\sqrt{\alpha}\eta,0)(1-|\eta|^2)^{\frac{n-2}{2}}d\eta \\
&= n\omega_n\alpha^{1-n}\int\limits_{|x|^2\le\alpha} u(x,0)(\alpha-|x|^2)^{\frac{n-2}{2}}dx. \qquad (2.8)
\end{aligned}$$

Now, because the integrand in Equation 2.8 is bounded for all $|x|\le\alpha$, the above limit exists and we may extend the domain of I to include $\beta=0$. Similarly, we observe $\lim_{\alpha\to 0^+} I(\alpha,\beta)$ exists, and so we also extend I accordingly. We see now that by Equation 2.7, we have

$$I(\alpha,0)=I(0,\alpha)\quad\text{for all }\alpha>0. \qquad (2.9)$$

Finally, define the spherical means in x-space and t-space respectively, by

$$I_1(R)=\frac{1}{n\omega_n}\int\limits_{|x|=R} u(x,0)dS_x\quad\text{and}\quad I_2(R)=\frac{1}{n\omega_n}\int\limits_{|t|=R} u(0,t)dS_t. \qquad (2.10)$$

So when considering Equations 2.8 and 2.9 with $\alpha=R^2$, we find that for any fixed $R>0$

$$\begin{aligned}
0 &= I(\alpha,0)-I(0,\alpha) \\
&= \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int\limits_{|x|^2\le R^2} u(x,0)(R^2-|x|^2)^{\frac{n-2}{2}}dx \\
&\quad -\frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int\limits_{|t|^2\le R^2} u(0,t)(R^2-|t|^2)^{\frac{n-2}{2}}dt \\
&= \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\int\limits_{|x|=r} u(x,0)dS_x dr \\
&\quad -\frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\int\limits_{|t|=r} u(0,t)dS_t dr \\
&= \frac{n\omega_n^2 R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\left(I_1(r)-I_2(r)\right)dr.
\end{aligned}$$

Thus, since $r^{n-1}(R^2-r^2)^{\frac{n-2}{2}} > 0$ for all $0 < r < R$, it must be that $I_1(r) = I_2(r)$ for all $r > 0$ because $R > 0$ was considered arbitrarily.

Ultimately, although it has only been shown that the theorem holds for $x_0 = t_0 = 0$, because the differential equation is linear, the result holds under any translation of the points x_0 and t_0. □

Observe that the assumption $n = m$ in Asgeirsson's Theorem is artificial and that the result still holds if we are considering equations of the form $\triangle_x u = \triangle_t u$, where $n \neq m$. In such a case, we seek solutions independent of the neglected variables and the corresponding integration for the mean values is taken about x_0 and t_0 in $\max\{n, m\}$-dimensional space.

3 Solutions and Well-Posedness

3.1 One time dimension

Assuming $n = 2k$ or $n = 2k+1$, then because $u \in C^2$ it can be shown that for any $f \in C^{k+2}$ and $g \in C^{k+1}$ the following closed form solutions of $[(2.1), (2.2)]$ exist for all $n \geq 1$ [Sie10]. When $n = 1$ the solution is well-known, and given by D'Alembert's formula

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

Furthermore, defining $c_n = 1 \cdot 3 \cdot \cdots \cdot (2k-1)$, we have for all other odd values of n

$$\begin{aligned} u(x,t) = &\frac{1}{c_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} f(x+t\xi)dS_\xi\right) \\ &+ \frac{1}{c_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} g(x+t\xi)dS_\xi \end{aligned}$$

Alternatively for all even n, using the method of descent upon the odd solutions above, we arrive with

$$\begin{aligned} u(x,t) = &\frac{1}{c_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-2}\int_{|\xi|\leq 1} \frac{f(x+t\xi)}{\sqrt{1+|\xi|^2}}d\xi\right) \\ &+ \frac{1}{c_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-2}\int_{|\xi|\leq 1} \frac{g(x+t\xi)}{\sqrt{1+|\xi|^2}}d\xi. \end{aligned}$$

We see by above formulas that the domain of dependence of the solution at (x,t) for $t > 0$ is $\{x + t\xi : |\xi| \leq 1\}$. This allows us to conclude that because the functions f and g in Equation 2.2 have compact support, so must the solution u when the t-variable is considered fixed. In particular, for any $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$ there is some $R > 0$ such that

$$u(x,t) = 0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_{R+t}(0). \tag{3.1}$$

Accounting for the above considerations, it will now be shown that all solutions to $[(2.1), (2.2)]$ with $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$, are unique. In essence, this suffices to show that for all $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$, the IVP $[(2.1), (2.2)]$ is well-posed because the continuous dependence of $[(2.1), (2.2)]$ follows directly from the closed form of the solutions above.

Proof of Uniqueness: Let u_1 and u_2 both be solutions of $[(2.1), (2.2)]$. Then for $v = u_1 - u_2$, we have that since the wave equation is linear,

$$\begin{cases} v_{tt} = \triangle_x v & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = f(x) - f(x) = 0 & \text{for } x \in \mathbb{R}^n \\ v_t(x,0) = g(x) - g(x) = 0 & \text{for } x \in \mathbb{R}^n \end{cases} \tag{3.2}$$

i.e. v satisfies $[(2.1), (2.2)]$ for $f = g = 0$.

Consider now, the global energy function arising from physical considerations,

$$E(t) = \int_{\mathbb{R}^n} \left({v_t}^2 + |\nabla_x v|^2 \right) dx,$$

and observe that

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^n} \left(v_t v_{tt} + \sum_{i=1}^{n} v_{x_i} v_{x_i t} \right) dx \\ &= \int_{B_{R+t}(0)} \left(v_t v_{tt} + \nabla_x v \cdot \nabla_x v_t \right) dx \\ &= \int_{B_{R+t}(0)} \left(v_t v_{tt} - v_t \triangle_x v \right) dx + \int_{\partial B_{R+t}(0)} v_t \nabla_x v \cdot \hat{\nu} dS \\ &= \int_{B_{R+t}(0)} v_t \cdot 0 dx + 0 \\ &= 0, \end{aligned}$$

where the second and fourth equality follow from Equations 3.1 and 3.2, and the third follows from Green's first identity.

Thus, for all $t > 0$, $E(t) = E(0) = 0$ and so

$$\nabla_x v = 0 \quad \text{and} \quad v_t = 0 \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Thus v is constant, which by Equation 3.2 implies $v = 0$ and so the uniqueness condition is indeed satisfied. □

3.2 Many time dimensions

Courant and Hilbert's classic argument will now be constructed in order to show that $[(2.3), (2.4)]$ is ill-posed [CH62].

The problem of determining functions from their mean values: Consider the spherical mean for a radius r of a function $u = u(y,t)$ centered at $(y,0)$ in (y,t)-space:

$$M_u(y,r) = \frac{1}{n\omega_n} \int\limits_{|\xi|^2 + \tau^2 = r^2} u(y+\xi, \tau) dS = Q[u].$$

Observe that through the symmetry of the $t = 0$ coordinate, $Q[u]$ depends only upon the even part of u in the t-variable, mainly $\frac{1}{2}(u(y,t) + u(y,-t))$. We seek to determine $u(y,t) + u(y,-t)$ from a particular known $M_u(y,r)$, and so through incredible foresight, we define

$$N_u(y,r) = \int_0^r M_u(y,\rho) d\rho = \frac{1}{n\omega_n} \int\limits_{|\xi|^2 + \tau^2 \leq r^2} u(y+\xi, \tau) d\xi \, d\tau. \tag{3.3}$$

Although it may appear unmotivated at the moment, differentiating N_u with respect to any of the y_i variables gives

$$\begin{aligned}\frac{\partial}{\partial y_i}N_u(y,r) &= \frac{1}{n\omega_n}\int\limits_{|\xi|^2+\tau^2\leq r^2} u_{y_i}(y+\xi,\tau)\ d\xi\ d\tau \\ &= \frac{1}{n\omega_n}\int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)\hat{\nu_i}\ dS \\ &= \frac{1}{n\omega_n r}\int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)\xi_i\ dS,\end{aligned}$$

where the second and third equalities follow by the divergence theorem and because $\hat{\nu_i} = \frac{\xi_i}{r}$. Thus, observe that

$$\begin{aligned}Q[u(y,t)y_i] &= \frac{1}{n\omega_n}\int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)(y_i+\xi_i)dS \\ &= y_i M_u(y,r) + r\frac{\partial}{\partial y_i}N_u(y,r) \\ &= y_i M_u(y,r) + r\frac{\partial}{\partial y_i}\int_0^r M_u(y,\rho)d\rho \\ &= D_i M_u,\end{aligned}$$

where

$$D_i = (y_i + r\frac{\partial}{\partial y_i}\int_0^r \cdot\ d\rho)$$

is a linear operator on the functions $M_u(y,r)$.

Through linearity, we now see that given a polynomial $P:\mathbb{R}^n\to\mathbb{R}$, we have

$$Q[Pu] = P(D_1,\ldots,D_n)M_u$$

and so $M_{Pu}(y,r) = Q[Pu]$ will be known given that we have M_u.

Alternatively, we have

$$\begin{aligned}Q[Pu] &= \frac{1}{n\omega_n}\int\limits_{|\xi|^2+\tau^2= r^2} P(y+\xi)u(y+\xi,\tau)dS_{\xi,\tau} \\ &= \frac{1}{n\omega_n}\int\limits_{|y-\eta|^2+\tau^2= r^2} P(\eta)u(\eta,\tau)dS_{\eta,\tau} \\ &= \frac{1}{2n\omega_n}\int\limits_{|y-\eta|^2+\tau^2= r^2} P(\eta)(u(\eta,\tau)+u(\eta,-\tau))dS_{\eta,\tau},\end{aligned}$$

where in the above we have taken $\eta = y+\xi$ and considered that $Q[u]$, and hence $Q[Pu]$, depend only upon the even part of u in the t-variable.

Now, for $|y-\eta|^2+\tau^2 = r^2$, we consider $\tau\geq 0$. So, writing $\tau = \phi(\eta) = \sqrt{r^2-|\eta-y|^2}$ we get

$$\begin{aligned}dS_{\eta,\tau} &= \sqrt{1+|\nabla_\eta\phi|^2}d\eta \\ &= \frac{\sqrt{\tau^2+|\eta-y|^2}}{\tau}d\eta \\ &= \frac{r}{\tau}d\eta.\end{aligned}$$

Thus,

$$Q[Pu] = \frac{r}{2n\omega_n} \int\limits_{|y-\eta|^2 \le r^2} P(\eta)(u(\eta,\tau) + u(\eta,-\tau))\frac{d\eta}{\tau}. \tag{3.4}$$

Finally through the Stone-Weierstrass Theorem, because polynomials are dense in $C(\overline{B}_r(0), \mathbb{R})$ under the supremum norm, we have that the function $\frac{1}{\tau}(u(\eta,\tau) + u(\eta,-\tau))$ can be determined uniquely by $Q[Pu] = P(D_1, \ldots, D_n)M_u$. We then receive from this, a unique determination of the even part of $u(y,t)$ for $|y_0 - y|^2 + t^2 = r^2$.[1] The details of this unique determination result distract from the proof, see Theorem A.1 in the appendix.

Claim 3.1. Given any sufficiently small $\epsilon > 0$, any $y_0 \in \mathbb{R}^n$, and any $r_0 > 0$, the even part of u, $\frac{1}{2}(u(y,t) + u(y,-t))$, upon the sphere $|y_0 - y|^2 + t^2 \le r_0^2$ is determined uniquely by $M_u(y_0, r_0)$ upon the finite cylinder, $0 \le r < r_0$ and $|y - y_0| \le \epsilon$.

Proof. Observe that in order to calculate $D_i M_u = y_i M_u(y,r) + r\frac{\partial}{\partial y_i}\int_0^r M_u(y,\rho)d\rho$ for y_0 and r_0 we need only to know $M_u(y,r)$ in some neighbourhood of y_0 in y-space and for $0 \le r < r_0$. Without loss of generality, assume that this necessary neighbourhood is a ball of radius $\epsilon > 0$ centered at y_0. We now have that in order to calculate $Q[Pu]$, it is necessary only to know $M_u(y,r)$ for $0 \le r < r_0$ and $|y - y_0| \le \epsilon$.

Furthermore, recalling Equation 3.4 and noting that we are only considering $0 \le r < r_0$, we find that M_u in the above cylinder uniquely determines the even part of u in the entire solid sphere $|y_0 - y|^2 + t^2 \le r_0^2$. □

Ill-posedness: Before we continue, it is relevant to note that any solution of $[(2.3), (2.4)]$ will be even in the t-variable because the map $t \mapsto -t$ preserves (2.3).

Now recall $[(2.3), (2.4)]$, i.e.

$$\begin{cases} u_{tt} = (\triangle_x - \triangle_y)\, u & \text{for } (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^m \\ u(x,y,0) = f(x,y) & \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \\ u_t(x,y,0) = g(x,y) & \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \end{cases}$$

and let G be a domain $G \subset \mathbb{R}^n$, $\epsilon > 0$, and consider only $y \in B_\epsilon(y_0)$ and $x \in G$.

Consider a solution u to $[(2.3), (2.2)]$. Then for fixed x we note that our prescribed function f determines M_u over all spheres in (y,t)-space such that $(y,t) \in B_\epsilon(y_0) \times \{0\}$ and whose radius r_0 is still small enough so that $B_{r_0}(x) \subset G$.

Recalling the proof of Asgeirsson's Theorem we now arrive with the following two cases:

Case 1 $(m \ge n)$: By Asgeirsson's Theorem directly, we find

$$\frac{1}{m\omega_m} \int\limits_{|\zeta'|^2 = r_0^2} u(x+\zeta, y, 0)dS_{\zeta'} = \frac{1}{m\omega_m} \int\limits_{|\xi|^2 + \tau^2 = r_0^2} u(x, y+\xi, \tau)dS_{\xi,\tau},$$

where and $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\zeta' = (\zeta_1, \ldots, \zeta_n, \ldots, \zeta_m)$.

[1] Although the function $\frac{1}{\tau}(u(\eta,\tau) + u(\eta,-\tau))$ is not continuous when $\tau = 0$ (i.e. when $|\eta - y| = r$), it is continuous on any compact set inside $\overline{B}_r(0)$, and so essentially implying our conclusion [CH62].

Case 2 ($n \geq m$): Invoking Asgeirsson's Theorem again, we find

$$\begin{aligned}
&\frac{1}{n\omega_n} \int\limits_{|\zeta|^2=r_0^2} u(x+\zeta, y, 0) dS_\zeta \\
&= \frac{1}{n\omega_n} \int\limits_{|\xi'|^2+\tau^2=r_0^2} u(x, y+\xi, \tau) dS_{\xi',\tau} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr} \int\limits_{|\xi'|^2+\tau^2\leq r^2} u(x, y+\xi, \tau) d\xi' d\tau \right]_{r=r_0} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr} \int\limits_{|\xi|^2+\tau^2\leq r^2} u(x, y+\xi, \tau) \int\limits_{|\xi'-\xi|^2\leq r^2-|\xi|^2-\tau^2} d\xi' d\tau \right]_{r=r_0} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr} \omega_{n-m} \int\limits_{|\xi|^2+\tau^2\leq r^2} u(x, y+\xi, \tau)(r^2-|\xi|^2-\tau^2)^{\frac{n-m}{2}} d\xi d\tau \right]_{r=r_0} \\
&= \frac{(n-m)\omega_{n-m} r_0}{n\omega_n} \int\limits_{|\xi|^2+\tau^2\leq r_0^2} u(x, y+\xi, \tau)(r_0^2-|\xi|^2-\tau^2)^{\frac{n-m-2}{2}} d\xi d\tau,
\end{aligned}$$

where $\xi = (\xi_1, \ldots, \xi_{m-1})$ and $\xi' = (\xi_1, \ldots, \xi_{m-1}, \ldots, \xi_{n-1})$. So in this case, similar to the proof of Asgeirsson's Theorem, we find that for all $0 \leq r < r_0$, the above equation may be rewritten:

$$\frac{1}{n\omega_n} \int\limits_{|\zeta|^2=r^2} u(x+\zeta, y, 0) dS_\zeta = \frac{m(n-m)\omega_{n-m}\omega_m r}{n\omega_n} \int_0^r \rho^{n-m-1}(r^2-\rho^2)^{\frac{n-m-2}{2}} I(\rho) d\rho, \tag{3.5}$$

where

$$I(r) = \frac{1}{m\omega_m} \int\limits_{|\xi|^2+\tau^2=r^2} u(x, y+\xi, \tau) dS_{\xi,\tau}$$

is the spherical mean of u for a radius r, M_u, for a fixed $x \in G$ taken in (y,t)-space at the point $(x, y, 0)$. Now, because the left hand side of Equation 3.5 is determined by f, differentiating Equation 3.5 twice with respect to r gives us a first order differential equation for $I(r)$ which has a unique solution by the existence and uniqueness theorem for ordinary differential equations.

Given arbitrary n and m, it is necessary that we restrict $r_0 \geq 0$ to be small enough in order that $B_{r_0}(x) \subset G$, i.e. so that the integral

$$\int\limits_{|\zeta|^2=r_0^2} u(x+\zeta, y, 0) dS_\zeta$$

is well defined. With this considered, we have by our previous claim that the even function $\frac{1}{2}(u(x,y,t) + u(x,y,-t))$, and thus also u itself, is uniquely determined in the sphere $|y_0 - y|^2 + t^2 \leq r_0^2$ by it's mean value M_u in (y,t)-space such that $(y,t) \in B_\epsilon(y_0) \times \{0\}$. Furthermore, consider that M_u itself is equal to the same integral of the prescribed function $f(x,y)$, and so M_u is determined uniquely by $f(x,y)$.

In an analogous way, the function $\frac{1}{2}(u_t(x,y,t) + u_t(x,y,-t))$ is determined uniquely by $g(x,y)$ and from this we get that $u(x,t)$ is determined uniquely by f and g. In particular, we get that $u(x,y,0)$ is determined for its initial value $t = 0$ inside the sphere in y-space, $|y_0 - y|^2 \leq r_0^2$.

Thus, we have proven that *if the initial values of a solution u of [(2.3), (2.4)] are known for $x \in G$ and t in an arbitrarily small sphere $|y_0 - y|^2 \leq \epsilon^2$, then the initial values are uniquely determined everywhere in the larger sphere $|y_0 - y|^2 \leq r_0^2$, where r_0 is defined above* [CH62].

Hence, arbitrary initial conditions cannot be imposed upon the wave equation with multiple time dimensions. This is in violation of any general well-posedness condition analogous to $[(2.1), (2.2)]$ since the existence of solutions will fail if the initial conditions f and g are not properly prescribed.

4 Conclusion

A fundamental difference between the two initial value problems $[(2.1), (2.2)]$ and $[(2.3), (2.4)]$ has now been highlighted. This being that the first problem is well-posed, and so in loose terms, physically significant; while the second problem is ill-posed and so not likely physically significant.

A possible physical interpretation of this is due specifically to the fact that when considering $[(2.3), (2.4)]$, we are prescribing our initial conditions on a 'mixed hypersurface'—this being a hypersurface extending not only in space, but also in time [Wei]. Thus the characteristics where upon a possible solution would propagate are time-like in some directions and must agree with the time-like prescribed initial conditions. This would inhibit a certain knowledge of future conditions which is highly unphysical.

In the past two years notable advancements of the well-posedness in the wave equation with many times have been pursued by W. Craig and S. Weinstein [CW09]. Their methods outline a specific Fourier transform derived constraint which, when imposed upon the respective initial conditions, has illuminated a new set of conditions bringing well-posedness to the initial value problem $[(2.3), (2.4)]$. An addendum article is planned to explore this particular development.

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A Unique determination of functions from dense integral operators

Theorem A.1. Let $r > 0$ and assume $f \in C(\overline{B}_r(0), \mathbb{R})$. Then, if

$$\int_{|x| \leq r} P(x) f(x) dx$$

is known for all $P \in S \subset C(\overline{B}_r(0), \mathbb{R})$, where S is some subset dense in supremum norm, we have a unique determination of the function f.

Proof. Suppose we have two functions, $f_1, f_2 \in C(\overline{B}_r(0), \mathbb{R})$ which have the same values under the operation $\int\limits_{|x|<r} P(x) \cdot dx$ for any $P \in S$. Then

$$\int\limits_{|x|\leq r} P(x)(f_1 - f_2)dx = 0$$

for all $P \in S$. Effectively, in order to show $f_1 = f_2$, we only need to show that if $f \in C(\overline{B}_r(0), \mathbb{R})$ and for all $P \in S$

$$\int\limits_{|x|\leq r} P(x)f(x)dx = 0,$$

then f must be the zero function.

So suppose that this is the case, but that $f \neq 0$. Then, since f is continuous, f is measurable, so $\operatorname{sgn} f \in L^1(B_r(0))$, so by the density of $C(\mathbb{R}^n, \mathbb{R})$ in L^1 we can find a sequence from S, $P_n(x)_{n=1}^{\infty}$, converging uniformly to $\operatorname{sgn} f$. Then let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|P_n - \operatorname{sgn} f\|_\infty < \frac{\epsilon}{\|f\|_1}.$$

We see that

$$\begin{aligned}
\|f\|_1 &= \left| \int\limits_{|x|<r} |f|(x)dx \right| \\
&= \left| \int\limits_{|x|<r} (f(x) \cdot \operatorname{sgn} f - P_n(x)f(x))dx \right| \\
&\leq \int\limits_{|x|<r} |f(x) \cdot (\operatorname{sgn} f - P_n(x))|dx \\
&\leq \|f\|_1 \| \operatorname{sgn} f - P_n\|_\infty \\
&< \epsilon
\end{aligned}$$

by Holders inequality. Since ϵ is arbitrary this is a contradiction, so $f = 0$. □