De Broglie-Bohm and Feynman Path Integrals

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Abstract: The de Broglie-Bohm theory offers what is arguably the clearest and most conceptually coherent formulation of nonrelativistic quantum mechanics known today. It not only renders entirely unnecessary all of the unresolved paradoxes at the heart of orthodox quantum theory, but moreover, it provides the simplest imaginable explanation for its entire (phenomenologically successful) mathematical formalism. All this, with only one modest requirement: the inclusion of precise particle positions as part of a complete quantum mechanical description. In this paper, we propose an alternative proof to a little known result—what we shall refer to as the de Broglie-Bohm path integral. Furthermore, we will show explicitly how the more famous Feynman path integral emerges and is, in fact, best understood as a consequence thereof.

1 Introduction

Despite its enormous success in the prediction of experimental regularities, orthodox quantum theory [Dir58, vN55] has been plagued, ever since its inception, by a wide range of conceptual difficulties. The most famous of these has come to be known as the measurement problem [Bel87]. That is, namely, the question of why a theory of fundamental physics should give axiomatic significance to such human-centered concepts as 'observers,' and why should it not be possible to understand 'measurement' as a particular case of some general evolution process rather than need to allocate a special postulate for it [dM02].

While many still regard the measurement problem as the principal foundational issue at the heart of orthodox quantum theory, it is perhaps best understood merely as a manifestation of a more fundamental conceptual inadequacy: that is, simply put, its inability to provide a clear notion of what quantum mechanics is actually *about*.

The central claim made by orthodox quantum theory is that the wavefunction provides a *complete* description of a quantum system. This would seem to entail that quantum mechanics is, fundamentally, about wavefunctions. However, accepting such a proposition faces us with an inescapable problem, most eloquently formulated by Schrödinger within the context of his famous cat paradox: namely, the question of what it actually means for an object to literally exist in a superposition of eigenstates of a measurement operator [Sch35]. The failure of orthodox quantum theory to offer any sort of coherent resolution to concerns of this sort is largely the reason for which it has continually remained so ambiguous and obscure.

Taking all of this into account, it may come as a surprise that very little is needed in order to essentially do away with all of the above perplexities. The de Broglie-Bohm theory (sometimes referred to as Bohmian mechanics or pilot-wave theory), proposed by Louis de Broglie in 1927 [dB27] and extended by David Bohm in 1952 [Boh52], accomplishes just that, in the simplest imaginable way. If, instead of wavefunctions, quantum mechanics really describes the behaviour of electrons and other elementary particles, then it seems that quantum mechanics should be, fundamentally, about particles in motion. As such, the essential insight of de Broglie-Bohm theory is that, in addition to the wavefunction, the description of a quantum system should also include its configuration—that is to say, the *precise* positions of all the particles at all times.

What naturally results from this is a deterministic quantum theory of particle trajectories—one which, as we shall see, not only accommodates, but provides the simplest known *explanation* for the quantum formalism (including the Born rule, the Heisenberg uncertainty relation, the representation of dynamical variables as Hilbert space operators, and so on).

This paper is organized as follows. Section 2 details the historical development of de Broglie-Bohm theory, and then Section 3 provides an exposition of its basic axioms. Section 4 is devoted to Bohm's second-order formulation of the theory in terms of the so-called quantum potential. In Section 5, we discuss, in broad outline, the notion of quantum equilibrium and the empirical equivalence between de Broglie-Bohm theory and orthodox quantum theory, while Section 6 briefly addresses some of the main criticisms which have been raised against the former. Then, in Section 7, we propose our alternative proof of the de Broglie-Bohm path integral, and in Section 8 we show how this naturally leads to the formulation of the Feynman path integral. Finally, Section 9 concludes the paper.

2 Historical Perspectives

At the 1927 Solvay conference in Brussels—perhaps the most important meeting in the history of quantum theory—Louis de Broglie presented what he called 'the new dynamics of quanta' [BV09]. This, the culmination of his independent work beginning in 1923 [dB23] and supplemented by Schrödinger's discoveries in 1926 [Sch26], amounted to the basic axioms of what is now referred to as de Broglie-Bohm theory.

De Broglie's fundamental insight was the following. In considering diffraction (the apparent bending of light when passing through narrow slits) which experiments had then demonstrated for X-rays, he argued [Val09] that this sort of phenomenon should not be understood—as in the classical picture—to be a wave-like manifestation of light itself. Rather, it should be viewed simply as evidence that photons do not always move in a straight line in empty space. What this amounts to, then, is a failure of Newton's first law, thereby requiring an entirely new form of dynamics—one that is based on *velocity*, not acceleration.

By the time he wrote his 1924 doctoral thesis [dB24], de Broglie had discovered precisely that: a unified dynamical theory of particles and waves, one where the latter guide the former along trajectories via a first-order law of motion (what is now referred to as the guiding equation). Moreover, this 'new dynamics' led him to making the extraordinary prediction [Val09] that material bodies (such as electrons) would also undergo diffraction, just like photons. (This was confirmed a few years later by Clinton Davisson and Lester Germer in their experiments on the scattering of electrons by crystals and, consequently, earned de Broglie the 1929 Nobel Prize for Physics.)

Inspired by this work, it was Schrödinger who, in 1926, developed the celebrated equation (now bearing his name) for de Broglie's waves. (This, despite the fact that Schrödinger dropped the trajectories from the theory, and considered only the waves.) Meanwhile, de Broglie unsuccessfully tried to derive the guiding equation from a still deeper theory, but by 1927, contented himself with presenting it as a provisional axiom of his 'new dynamics' [dB27].

Contrary to common misconceptions, de Broglie's theory was no less extensively discussed at the Solvay conference than any of the alternative proposals, with support expressed by Einstein and Brillouin [BV09]. Nevertheless, de Broglie abandoned his ideas a few years later. A large part of the reason for this—again, contrary to misconceptions—had to do with the fact that he could not construct an analysis of the quantum measurement process from his theory (and thereby show explicitly how it reproduces all of the successful empirical predictions of orthodox quantum theory). That problem was resolved in 1952 by Bohm [Boh52].

What then became known as the de Broglie-Bohm theory was essentially ignored for most of the second half of the last century, despite ceaseless efforts by John Bell to promote it as "an antidote to the prevailing complacency" [Bel87]. However, the situation has changed considerably over the course of the past two decades. The early 1990s saw the publication of the first textbooks to present quantum mechanics from this perspective [BH93, Hol93], and the number of physicists working on it has since increased substantially. The de Broglie-Bohm theory is now accepted as an alternative—if little used—formulation of quantum mechanics.

Now we are ready to state—in full mathematical detail—the theory's axioms, which are typically formulated in more or less the same way as de Broglie presented them in 1927.

3 The Axioms of de Broglie-Bohm Theory

For any system of N nonrelativistic spinless particles, there exists an associated wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ belonging to the set of complex-valued functions which are square-integrable on the configuration space

$$\Omega = \left\{ q = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \middle| \, \mathbf{q}_j = (q_{j1}, q_{j2}, q_{j3}) \in \mathbb{R}^3 \right\} \subseteq \mathbb{R}^{3N}.$$

The de Broglie-Bohm theory postulates [BH93, Hol93, DT09, BDDDZ95] that the system is *completely described* by its wavefunction $\psi(q,t)$ and its configuration $\mathcal{Q}(t) = (\mathbf{Q}_1(t), \ldots, \mathbf{Q}_N(t)) \in \Omega$ (where $\mathbf{Q}_j(t) \in \mathbb{R}^3$ is the position of the j-th particle).

The theory is defined by the following two axioms:

Axiom 1 (The Schrödinger Equation). The wavefunction evolves via

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \nabla_j^2 \psi + V\psi, \tag{3.1}$$

where $\hbar$ is the reduced Planck constant, m_j is the mass of the j-th particle, $V : \Omega \to \mathbb{R}$ is the potential, and $\nabla_j = \partial / \partial \boldsymbol{q}_j$.

Axiom 2 (The Guiding Equation). The position of the j-th particle is determined by the equation of motion

$$\frac{d\mathbf{Q}_j}{dt} = \frac{\hbar}{m_j} \Im \left(\frac{\nabla_j \psi}{\psi} \right) \Bigg|_{\mathcal{Q}(t)}. \tag{3.2}$$

Equivalently, for any wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ there exists an associated velocity field $v^\psi = (\mathbf{v}_1^\psi, \ldots, \mathbf{v}_N^\psi) : \Omega \to \Omega$ defined according to

$$\mathbf{v}_j^\psi = \frac{\hbar}{m_j} \Im \left(\frac{\nabla_j \psi}{\psi} \right),$$

such that the configuration of the system follows

$$\frac{d\mathcal{Q}}{dt} = v^\psi \big(\mathcal{Q}(t) \big).$$

Equations 3.1 and 3.2 form a complete specification of the theory, giving rise to a clear and astonishingly simple picture of the evolution of a de Broglie-Bohm universe: particles (a concept taken in its most literal sense) move along trajectories (solutions of Equation 3.2) that are choreographed by the wavefunction (the solution of Equation 3.1).

We stress that there is no need—and, indeed, no room—for any further axioms. In particular, no 'measurement' axioms (a concept central to orthodox quantum theory) are necessary. In de Broglie-Bohm theory, there is no special role for 'the observer,' nothing to elevate the status of a 'measurement' above that of any other physical process.

Furthermore, de Broglie-Bohm theory is deterministic. While the issue of determinism was never a primary motivation for its creation [BH93], it so happens that the simplest possible quantum theory of particle motion has this feature.

Now, before addressing the non-trivial question of how such a theory could possibly reproduce all of the well-known probabilistic results of orthodox quantum theory (that will be left for Section 5), we consider an even more immediate concern: supposing that we indeed accept trajectories as indispensable to a complete quantum mechanical description, what reason is there for the particular choice of Equation 3.2 as the basic law of motion? As we have seen, de Broglie eventually recognized no alternative to simply giving the guiding equation the same status as the Schrödinger equation—namely, that of an axiom with no more ultimate justification—and yet, remarkably, there are a number of different lines of reasoning by which the former can be shown to naturally emerge from the latter.

We now present one of the more common approaches to resolving this issue. Consider again a system of N particles with some wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$. With a suitable choice of a phase function $S : \Omega \times \mathbb{R} \to \mathbb{R}$, we can rewrite the wavefunction in the polar form

$$\psi = |\psi| \exp\left(\frac{i}{\hbar} S\right). \tag{3.3}$$

Inserting Equation 3.3 into the Schrödinger equation, we obtain, after separating real and imaginary parts,

$$\frac{\partial |\psi|^2}{\partial t} + \sum_{j=1}^{N} \nabla_j \cdot \left(|\psi|^2 \frac{\nabla_j S}{m_j}\right) = 0, \tag{3.4}$$

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} + V + \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \left(\frac{\nabla_j^2 |\psi|}{|\psi|}\right) = 0. \tag{3.5}$$

While Equation 3.4 is just the familiar continuity equation (expressing conservation of probability), we notice that Equation 3.5 is reminiscent of the following result from classical mechanics:

Theorem 3.1 (The Jacobi Theorem). In a system of N particles obeying classical mechanics, if $S : \Omega \times \mathbb{R} \to \mathbb{R}$ is any complete solution of the partial differential equation

$$\frac{\partial S}{\partial t} + \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} + V = 0, \tag{3.6}$$

known as the Hamilton-Jacobi equation, then the equation of motion

$$\frac{d\mathbf{Q}_j}{dt} = \left.\frac{\nabla_j S}{m_j}\right|_{\mathcal{Q}(t)} \tag{3.7}$$

can solved to yield the system's configuration $\mathcal{Q}(t) = (\mathbf{Q}_1(t), \ldots, \mathbf{Q}_N(t)) \in \Omega$, where $\mathbf{Q}_j(t) \in \mathbb{R}^3$ is the position of the j-th particle.

Proof. See Takhtajan's book [Tak08]. □

Now, observe that Equation 3.6 is identical to Equation 3.5 except for the fourth term on the left hand side of the latter. We will address the precise meaning of this quantity in the next section. For now we can immediately deduce, as it is on the order of the square of the Planck constant, that it will be negligible at macroscopic distances (i.e. where classical mechanics is presumed to apply). That is to say, we can interpret Equation 3.6 as the classical limit of Equation 3.5, which is consequently referred to as the *quantum* Hamilton-Jacobi equation. But if that is the case, then the function S in Equation 3.7 must be the same as the phase function from Equation 3.3. Combining these two relations gives us, after a bit of algebra, Equation 3.2 from Axiom 2. (Another approach for obtaining this result, it turns out, is just to seek the simplest first-order equation of motion dependent on the wavefunction that preserves Galilean and time-reversal invariance [DGZ92].)

Remark 3.2: The above considerations give us many different ways of writing the components of the velocity field v^ψ,

$$\mathbf{v}_j^\psi = \frac{\hbar}{m_j} \Im\left(\frac{\nabla_j \psi}{\psi}\right) = \frac{\nabla_j S}{m_j} = \frac{\boldsymbol{\mathcal{J}}_j^\psi}{|\psi|^2}, \tag{3.8}$$

where the reader familiar with orthodox quantum theory will recognize the the quantum probability current $\mathcal{J}^\psi = (\boldsymbol{\mathcal{J}}_1^\psi, \boldsymbol{\mathcal{J}}_2^\psi, \ldots, \boldsymbol{\mathcal{J}}_N^\psi) : \Omega \to \Omega$ defined according to $\boldsymbol{\mathcal{J}}_j^\psi = (i\hbar/2m_j)\left(\psi \nabla_j \psi^* - \psi^* \nabla_j \psi\right)$.

4 The Quantum Potential and the Bohm Equation of Motion

It is sometimes useful to reformulate de Broglie-Bohm theory in terms of the following:

Definition 4.1 (The Quantum Potential). For an N-particle system with wavefunction $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$, the quantum potential is defined to be

$$V_q = \sum_{j=1}^{N} \frac{-\hbar^2}{2m_j} \left(\frac{\nabla_j^2 |\psi|}{|\psi|} \right). \tag{4.1}$$

With this definition, we may suggestively write:

Theorem 4.1 (The Bohm Equation of Motion). The trajectory of the j-th particle is determined by

$$m_j \frac{d^2 \mathbf{Q}_j}{dt^2} = -\nabla_j \big(V + V_q\big)\big|_{\mathcal{Q}(t)}. \tag{4.2}$$

Proof. The above follows directly from taking the time derivative of both sides of Equation 3.2. □

We note here that, in contrast to de Broglie, Bohm regarded Equation 4.2 as the theory's fundamental equation of motion, with Equation 3.2 taken simply as a constraint [BH93]. The justification, often seen as somewhat contrived, proceeds along the following lines: since Equation 3.6 is interpreted as being the classical limit of Equation 3.5, the meaning of the term V_q in the latter is necessarily that of an additional potential to the classical potential V; from this, the quasi-Newtonian law of motion in Equation 4.2 logically follows.

However, it is rather unnatural to recast de Broglie-Bohm theory in this way, i.e. as a second-order theory. Indeed, as we have seen, de Broglie's essential motivation for its formulation had been the need for a new (non-Newtonian) *velocity-based* dynamics, the essence of which is captured by the guiding equation. Nevertheless, the Bohm equation of motion is a useful result insofar as it immediately allows us to draw the following observation:

Remark 4.2: In the macroscopic limit, $V_q \to 0$ and so the Bohm equation of motion reduces to Newton's second law i.e. de Broglie-Bohm theory reduces to classical mechanics.

Furthermore, we will later on need to make use of the so-called Lagrangian function. This is a notion that originates from classical mechanics, where it is defined as follows:

Definition 4.2 (The Classical Lagrangian). The Lagrangian function in classical mechanics is defined as

$$\mathcal{L}_c = \sum_{j=1}^{N} \frac{1}{2} m_j \left\| \mathbf{v}_j^{\psi} \right\|^2 - V, \tag{4.3}$$

where $\mathbf{v}_j^{\psi}$ is the velocity of the j-th particle.

In fact, it can be shown that knowing the Lagrangian function is sufficient for obtaining a complete description of any system in classical mechanics (insofar as all of the equations of motion for that system can be derived directly therefrom). Therefore, it is possible and often quite convenient to formulate all of classical mechanics in terms of this quantity.

Now, by analogy, we can also define:

Definition 4.3 (The Quantum Lagrangian). Inspired by the classical Lagrangian function, we define the quantum Lagrangian to be

$$\mathcal{L}_q = \sum_{j=1}^{N} \frac{1}{2} m_j \left\| \mathbf{v}_j^{\psi} \right\|^2 - (V + V_q), \tag{4.4}$$

with $\mathbf{v}_j^{\psi}$ and V_q given by Equations 3.8 and 4.1 respectively.

Just as in the classical case, it is certainly possible to formulate de Broglie-Bohm theory in terms of Equation 4.4. However, we do not wish to enter into the details of that here; the usefulness of the above two definitions will, for our purposes, become apparent in Sections 7 and 8.

For the moment, however, we turn our attention to the question of how the notions of randomness and uncertainty enter into de Broglie-Bohm theory.

5 Quantum Equilibrium and the Emergence of Absolute Uncertainty

As noted before, de Broglie-Bohm theory describes a universe which is completely deterministic. Yet, it is well known that in our universe, all quantum phenomena appear to systematically yield random outcomes. Orthodox quantum theory accounts for this as follows. A quantum system, therein presumed to be completely described by its wavefunction $\psi(q,t)$, evolves (deterministically) according to the Schrödinger equation, *until* such a time as a 'measurement' is performed. Then, the wavefunction randomly 'collapses' onto one of all the possible outcomes, the statistical distribution function for which is given by $\rho(q,t) = |\psi(q,t)|^2$. This is the so-called Born rule.

Now, while Section 1 has already touched upon the difficulties that arise from ascribing axiomatic status to the Born rule, it does nevertheless—so far as we know—prescribe the correct probabilities for the outcomes of quantum measurements. Put differently, our world indeed looks to us random, with probability distribution $\rho = |\psi|^2$. However—and we stress this yet again—it is a randomness that manifests itself *at no times other than* during 'measurements'. Therefore, rather than draw from this the (all too common) conclusion that there is some sort of fundamental element of chance to how our universe works, it seems far more natural to interpret the notion of probability (associated with the outcomes of measurements) as an emergent feature of some underlying dynamics—a description of which is precisely what de Broglie-Bohm theory has to offer.

The axiomatic status of the Born rule in orthodox quantum theory seems even more bewildering when we consider that this concept of emergent probability is nothing new in physics. Indeed, the same notion lies at the basis of classical statistical mechanics, where the underlying (deterministic) Newtonian dynamics gives rise to the apparently random behaviour of large systems (such as, for instance, the famous Maxwell-Boltzmann distribution of molecular velocities in a gas).

So, if we expect quantum probabilities to likewise have a dynamical origin and the correct account thereof to be that which is given by de Broglie-Bohm theory, then the question which remains to be addressed is this: how exactly does the $\rho = |\psi|^2$ distribution arise therefrom? It turns out that the solution to this problem is very non-trivial; its satisfactory understanding requires exhaustive analysis [DGZ92, Val01, VW05], to which we can provide here only an outline.

The discussion hinges upon the notion of equilibrium as a general feature of dynamical systems. Broadly speaking, a system is said to be in a statistical state of equilibrium (for example, thermodynamic equilibrium in classical statistical mechanics) once it has reached some measure of stationarity—that is to say, a time-invariant distribution. It turns out that in de Broglie-Bohm theory, $\rho = |\psi|^2$ fits precisely this description, and in this context is referred to as being *equivariant*. Formally, this can be stated as:

Theorem 5.1 (The Equivariance Theorem). If $\rho(q,t_0) = |\psi(q,t_0)|^2$ for some initial time t_0, then $\rho(q,t) = |\psi(q,t)|^2$, $\forall t$.

Proof. This follows trivially from Equation 3.4. □

However, even more can be said:

Theorem 5.2 (The Uniqueness of Quantum Equilibrium). $\rho = |\psi|^2$ is the unique equivariant distribution that is also a local functional of ψ.

Proof. See Goldstein and Struyve's paper [GS07]. □

The above theorems suggest that if indeed equivariance is the required condition for quantum equilibrium—that is, an equilibrium state of the system relative to the wavefunction—then its only possible measure is the $\rho = |\psi|^2$ distribution, thus leading to:

Claim 5.1 (The Quantum Equilibrium Hypothesis). For a system in quantum equilibrium having wavefunction ψ, the statistical distribution of its coordinates is given by

$$\rho\,(q,t) = |\psi\,(q,t)|^2 \,. \tag{5.1}$$

This result—whose full justification [DGZ92, Val01, VW05] is, once again, beyond the scope of this paper—gives rise to the notion of absolute uncertainty (which is essentially equivalent to the Born rule): that it is impossible to know more about the configuration of a system beyond that which Equation 5.1 allows. In other words, a universe governed by (deterministic) de Broglie-Bohm dynamics evolves so that, when in quantum equilibrium, the *appearance* of randomness emerges (in much the same way as it does, when in thermodynamic equilibrium, for one governed by classical statistical mechanics).

Now, the natural question that arises from the above argument is this: can there exist systems that are *not* in quantum equilibrium, i.e. for which $\rho \neq |\psi|^2$? While orthodox quantum theory deems this inadmissible (it being an explicit violation of the Born rule), there is nothing in de Broglie-Bohm theory preventing such a possibility. However, numerical simulations [VW05] have demonstrated the effective relaxation of such systems to the equilibrium distribution $\rho = |\psi|^2$ (where their discrepancy therewith was found to decrease approximately exponentially over time).

Moreover, it has recently been proposed [Val07, Val10] that our universe began with a non-equilibrium distribution $\rho \neq |\psi|^2$ (as there is, a priori, no reason to favour an initial equilibrium state), with sufficient time to then relax to the $\rho = |\psi|^2$ distribution we observe today. This avenue of exploration is still in its early stages, but (if the above claim turns out to be true) may soon offer the first possibility of an empirical distinction between de Broglie-Bohm theory and orthodox quantum theory via observations of the early universe.

Otherwise, in our present universe (i.e. which is presumed to have long reached quantum equilibrium), Claim 5.1 is sufficient to guarantee the emergence of the usual mathematical formalism of quantum mechanics—operators as observables and all the rest—from de Broglie-Bohm theory. (The reader is referred to Dürr, et al. [DGZ04] for an analysis of how this comes about in detail.) Therefore, de Broglie-Bohm theory not only reproduces all of the familiar predictions of orthodox quantum theory, but it does so in such a manner as to eliminate the measurement problem (see Goldstein, et al. [GTZ10] for a simple treatment of this) as well as any dubious notions regarding what the reality of a quantum system might actually be.

6 Criticisms of de Broglie-Bohm Theory

We turn briefly to a consideration of the main criticisms presently facing de Broglie-Bohm theory. Perhaps the most serious among these is that there are well-known issues surrounding the attempts to find for it a Lorentz invariant extension. What makes this problematic is the nonlocality of the theory manifested through the guiding equation, i.e. the fact that the trajectory of each particle depends *instantaneously* on the positions of *all* the particles in the system. (In fact, Bell showed [Bel64] that nonlocality is a necessary feature of quantum mechanics under *any* interpretation, and considered it "a merit of the de Broglie-Bohm version to bring this out so explicitly that it cannot be ignored" [Bel87].) Despite the obvious problems that this imposes, a number of limited approaches towards a relativistic generalization of the theory have so far been proposed [BH93, BDGZ96, DGMBZ99], making a complete rejection of de Broglie-Bohm theory on this basis rather premature. (There are, on the other hand, interesting arguments to the effect that, due to the first-order structure of the theory, it is an interpretational mistake to even conceive of Lorentz invariance as a fundamental symmetry; the search for it, then, would be essentially misguided [Val97].) Furthermore, contrary to common suspicions, an extension of de Broglie-Bohm theory to quantum field theories does turn out to be possible as well [Bel87, DGTZ04].

Most of the other criticisms which have been voiced against this theory (for example, that it is 'complicated' or 'contrived,' that it constitutes a regression back to classical physics, that it does not postulate the quantum equilibrium hypothesis, and other such assertions) lack much of a compelling basis and will not be discussed any further here. (For detailed responses to objections of this sort, the reader is referred to articles by Passon and Kiessling [Pas05, Kie10].)

7 The de Broglie-Bohm Path Integral

A fundamental problem in quantum mechanics is the propagation of the wavefunction: that is, given the wavefunction $\psi(q,t)$ at some initial time t, we desire to calculate $\psi(q',t')$, $\forall q' \in \Omega$ at any future time t', assuming its evolution to be governed by the Schrödinger equation, i.e. Equation 3.1. It turns out that in certain simple situations (such as when $V = 0$), this is something that can be resolved without much effort [Tak08]. However, for an arbitrary potential V, the problem becomes non-trivial and so, in order to formulate a solution, we shall have to construct so-called quantum mechanical path integrals. That is the aim of this and the next section; in particular, we will first consider the solution from the perspective of de Broglie-Bohm theory, and then we will show how to obtain from it the usual solution from orthodox quantum theory.

The propagation of the wavefunction in the context of de Broglie-Bohm theory has recently been considered and achieved via what we shall call the de Broglie-Bohm path integral [AG03]. Our aim here is to provide for it an alternative proof.

Consider again the wavefunction in polar form, i.e. Equation 3.3. In order to propagate it in this representation, we shall propagate separately its modulus as well as its phase function. Between any $q \in \Omega, t \in \mathbb{R}$ and any $q' \in \Omega, t' \in \mathbb{R}$ along de Broglie-Bohm trajectories, we obtain the following:

Lemma 7.1. The wavefunction modulus is propagated according to

$$|\psi(q',t')| = \left[\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\int_{t}^{t'} \nabla\cdot\mathbf{v}_j^{\psi}dt\right)\right]|\psi(q,t)|, \tag{7.1}$$

where $\mathbf{v}_j^{\psi}$ is given by Equation 3.8.

Proof. The continuity equation, i.e. Equation 3.4, can be rewritten as

$$\frac{\partial|\psi|}{\partial t} = -\frac{|\psi|}{2}\sum_{j=1}^{N}\nabla_j\cdot\mathbf{v}_j^{\psi}.$$

Now, separating and integrating along our de Broglie-Bohm paths, we get

$$\ln\left|\frac{\psi(q',t')}{\psi(q,t)}\right| = -\frac{1}{2}\sum_{j=1}^{N}\int_{t}^{t'}\nabla\cdot\mathbf{v}_j^{\psi}dt,$$

from which Equation 7.1 immediately follows. □

Lemma 7.2. The phase function is propagated according to

$$S(q',t') = S(q,t) + \int_{t}^{t'}\mathcal{L}_q dt, \tag{7.2}$$

where the quantum Lagrangian $\mathcal{L}_q$ is given by Equation 4.4.

Proof. Consider the time rate of change of the function S (i.e. the total derivative dS/dt) along the trajectories. By the chain rule, we know that this can be decomposed into contributions from the motion of all the particles (along their respective paths) plus its time rate of change at a fixed configuration (i.e. the partial derivative $\partial S/\partial t$); that is to say,

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\partial S}{\partial \boldsymbol{q}_j} \cdot \frac{d\boldsymbol{q}_j}{dt} + \frac{\partial S}{\partial t}.$$

As we are evaluating dS/dt along de Broglie-Bohm paths, the terms $d\boldsymbol{q}_j/dt$ are simply the components of the velocity field given by Equation 3.8. This means that we can write $d\boldsymbol{q}_j/dt = \nabla_j S/m_j$, and therefore,

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{m_j} + \frac{\partial S}{\partial t}.$$

Now, using the expression for $\partial S/\partial t$ given by the quantum Hamilton-Jacobi equation, i.e. Equation 3.5, we have

$$\frac{dS}{dt} = \sum_{j=1}^{N} \frac{\|\nabla_j S\|^2}{2m_j} - (V + V_q) = \mathcal{L}_q.$$

In other words, the total time derivative of S along de Broglie-Bohm paths is simply the quantum Lagrangian. Finally, we can substitute this into

$$S(q', t') = S(q, t) + \int_t^{t'} \left(\frac{dS}{dt}\right) dt$$

to obtain precisely Equation 7.2. □

Now we put together Lemmas 7.1 and 7.2, thereby allowing us to finally prove the main result of this paper:

Theorem 7.1 (The de Broglie-Bohm Path Integral). The wavefunction is propagated via the de Broglie-Bohm path integral according to

$$\psi(q', t') = \exp\left[\int_t^{t'} \left(\frac{i}{\hbar}\mathcal{L}_q - \frac{1}{2}\sum_{j=1}^{N} \nabla_j \cdot \mathbf{v}_j^{\psi}\right) dt\right] \psi(q, t).$$

Proof. By Lemma 7.2, we have that

$$\exp\left[\frac{i}{\hbar}S(q', t')\right] = \exp\left(\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt\right) \exp\left[\frac{i}{\hbar}S(q, t)\right].$$

Combining the above with Lemma 7.1, we get

$$|\psi(q', t')| \exp\left[\frac{i}{\hbar}S(q', t')\right] = \exp\left(\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{N} \int_t^{t'} \nabla \cdot \mathbf{v}_j^{\psi} dt\right) |\psi(q, t)| \exp\left[\frac{i}{\hbar}S(q, t)\right].$$

Recalling the polar form from Equation 3.3, this simply means that

$$\psi(q', t') = \exp\left[\int_t^{t'} \frac{i}{\hbar}\mathcal{L}_q dt - \frac{1}{2}\sum_{j=1}^{N} \int_t^{t'} \nabla \cdot \mathbf{v}_j^{\psi} dt\right] \psi(q, t),$$

as desired. □

8 The Feynman Path Integral from Theorem 7.1

We have thus far seen how it is possible to construct a path integral formalism for the de Broglie-Bohm theory—one which has, at its heart, the notion of precise particle paths. As mentioned before, orthodox quantum theory provides its own solution for propagating the wavefunction—namely, the Feynman path integral [Fey48] (to be stated explicitly below) which, moreover, also turns out to have useful applicability in quantum field theory (for instance, as a method of quantizing the electromagnetic field [Fey49, Fey50]).

Now, despite the fact that de Broglie-Bohm path integrals and Feynman path integrals originate from strikingly different conceptual bases, it still seems natural to pose the question: what exactly does one have to do (if anything at all) with the other? The answer turns out to be quite remarkable: the latter can be constructed directly from an analysis of the former.

Although a "heuristic argument" for this result has already been put forth in [AG03], our aim here is to prove it a bit more carefully and rigorously. For the sake of simplicity, we shall do this for the single particle case; Theorem 7.1 then reduces to

$$\psi\left(\mathbf{q}', t'\right) = \exp\left[\int_t^{t'} \left(\frac{i}{\hbar}\mathcal{L}_q - \frac{1}{2}\nabla \cdot \mathbf{v}^{\psi}\right) dt\right] \psi\left(\mathbf{q}, t\right), \tag{8.1}$$

where the integration is performed over the de Broglie-Bohm path taken by the particle from $(\mathbf{q}, t)$ to $(\mathbf{q}', t')$.

We will soon see that the propagation of the wavefunction as prescribed by the Feynman path integral is naturally expressed in terms of a function known as the propagator. So, before proceeding further, we give its definition:

Definition 8.1 (The Propagator). The propagator $K\left(\mathbf{q}', t'; \mathbf{q}, t\right)$ of the wavefunction ψ is defined such that

$$\psi\left(\mathbf{q}', t'\right) = \int_{\mathbb{R}^3} K\left(\mathbf{q}', t'; \mathbf{q}, t\right) \psi\left(\mathbf{q}, t\right) d^3\mathbf{q}.$$

Remark 8.1: In orthodox quantum theory, the standard meaning assigned to the propagator (or 'complex probability amplitude') is the following: the function $|K\left(\mathbf{q}', t'; \mathbf{q}, t\right)|^2$ is the conditional probability distribution of finding the particle at point $\mathbf{q}'$ at time t' provided that it was at point $\mathbf{q}$ at time t [Tak08].

Furthermore, we shall need to make use of the following result:

Theorem 8.2 (The Free Particle Wavefunction). For the free particle, i.e. when $V = 0$, the solution to the Schrödinger equation is

$$\psi\left(\mathbf{q}, t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int_{\mathbb{R}^3} \exp\left(\frac{i}{\hbar}\mathbf{p} \cdot \mathbf{q}\right) \widehat{\psi}\left(\mathbf{p}, t\right) d^3\mathbf{p}, \tag{8.2}$$

where $\widehat{\psi} = \mathcal{F}\left(\psi\right)$ is the Fourier transform of the wavefunction.

Proof. See Berndl, et al. [BDGZ96]. □

Now we are ready to construct the Feynman path integral:

Theorem 8.3 (The Feynman Path Integral). Let $[t, t'] \subset \mathbb{R}$ be divided into n infinitesimal slices $\{[t_k, t_{k+1}] | \, t_{k+1} - t_k = \Delta t, \forall k\}_{k=0}^{n-1}$ with $t_0 = t$ and $t_n = t'$; furthermore, let each t_k correspond to position coordinate $\mathbf{q}_k \in \mathbb{R}^3$ with $\mathbf{q}_0 = \mathbf{q}$ and $\mathbf{q}_n = \mathbf{q}'$. Then, the propagator for $\psi\left(\mathbf{q}, t\right)$ is

$$K\left(\mathbf{q}', t'; \mathbf{q}, t\right) = \lim_{n\to\infty} \int \cdots \int_{\mathbb{R}^{3(n-1)}} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{3}{2}n} \exp\left(\frac{i}{\hbar}\int_t^{t'} \mathcal{L}_c dt\right) \prod_{k=1}^{n-1} d^3\mathbf{q}_k, \tag{8.3}$$

where the classical Lagrangian $\mathcal{L}_c$ is given by Equation 4.3.

Proof. Consider the free particle solution from Theorem 8.2 on $[t_k, t_{k+1}], \forall k$. Observe that, since the time interval is taken to be infinitesimal, this solution is still valid even if the potential V is arbitrary (i.e. not necessarily zero). For this wavefunction, then, the guiding equation can be shown to yield a constant $m\mathbf{v}^{\psi} = \mathbf{p}$ for the particle's momentum. This immediately implies that $\nabla \cdot \mathbf{v}^{\psi} = 0$; furthermore, since $V_q = 0$ as well, we will have $\mathcal{L}_q = \mathcal{L}_c$ (i.e. the quantum Lagrangian is the same as the classical one). Hence,

$$\int_{t_k}^{t_{k+1}} \mathcal{L}_q dt = \int_{t_k}^{t_{k+1}} \left(\frac{\|\mathbf{p}\|^2}{2m} - V \right) dt = \left(\frac{m}{2} \left\| \frac{\mathbf{q}_{k+1} - \mathbf{q}_k}{\Delta t} \right\|^2 - V \right) \Delta t,$$

and so Equation 8.1 yields

$$\psi\left(\mathbf{q}_{k+1}, t_{k+1}\right) = \left\{ \exp\left[\frac{i}{\hbar} \left(\frac{m}{2} \frac{\|\mathbf{q}_{k+1} - \mathbf{q}_k\|^2}{\Delta t} - V \Delta t \right) \right] \right\} \psi\left(\mathbf{q}_k, t_k\right).$$

Now, it can be shown (see [GTZ10, BDGZ96]) that the propagator for a wavefunction in the above form (with $\psi\left(\mathbf{q}_k, t_k\right)$ given by Equation 8.2) is

$$K\left(\mathbf{q}_{k+1}, t_{k+1}; \mathbf{q}_k, t_k\right) = \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar} \left(\frac{m}{2} \frac{\|\mathbf{q}_{k+1} - \mathbf{q}_k\|^2}{\Delta t} - V \Delta t \right) \right]. \tag{8.4}$$

From here, we proceed as in all standard proofs of this theorem. The total propagator $K\left(\mathbf{q}', t'; \mathbf{q}, t\right)$ can be obtained via the integral representation

$$K\left(\mathbf{q}', t'; \mathbf{q}, t\right) = \int \cdots \int_{\mathbb{R}^{3(n-1)}} \prod_{k=0}^{n-1} K\left(\mathbf{q}_{k+1}, t_{k+1}; \mathbf{q}_k, t_k\right) \prod_{k=1}^{n-1} d^3\mathbf{q}_k.$$

Substituting Equation 8.4 into this gives

$$K\left(\mathbf{q}', t'; \mathbf{q}, t\right) = \int \cdots \int_{\mathbb{R}^{3(n-1)}} \left(\frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{3}{2}n} \exp\left[\sum_{k=0}^{n-1} \frac{i}{\hbar} \left(\frac{m}{2} \left\| \frac{\mathbf{q}_{k+1} - \mathbf{q}_k}{\Delta t} \right\|^2 - V \right) \Delta t \right] \prod_{k=1}^{n-1} d^3\mathbf{q}_k.$$

Taking the $n \to \infty$ limit, the summation in the exponential becomes an integral and we recover exactly Equation 8.3, as desired. □

9 Conclusion

The Feynman path integral is conventionally understood as a sum over all (infinite) possible paths connecting $(\mathbf{q}, t)$ and $(\mathbf{q}', t')$, each of these contributing with an amplitude found by integrating the classical Lagrangian. However, these paths are understood not be real paths, i.e. along which the particle actually moves. (Of course, in orthodox quantum theory, such a concept does not even exist). Rather, they are seen merely as mathematical tools useful for computing the evolution of the wavefunction. In this sense, the Feynman path integral is nothing more than a reformulation of the Schrödinger equation.

However, de Broglie-Bohm theory requires a bit more than this to make the quantum picture complete: namely, that the particle actually does move along one of the possible paths, in accordance with the guiding equation. As we have seen (via Equation 8.1), the evolution of the wavefunction can in this case be calculated, quite elegantly, by integrating the quantum Lagrangian along this one single path i.e. the particle's de Broglie-Bohm trajectory. It should then come as no surprise that the Feynman method of summing over all paths can be constructed with the de Broglie-Bohm theory at its basis.

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