The Wave Equation and Multi-Dimensional Time

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Abstract: The classical wave equation initial value problem in single and multiple time dimensions is posed and subsequently, the physical and mathematical basis of it is discussed. The Theorem of Asgeirsson is proved and applied to study the wave equation with multiple time dimensions. Further, with the assembly of work by Courant and Hilbert, the well-posedness of such problems is determined in detail.

1 Introduction

Physical theories or interpretations with multiple time dimensions are usually ignored or shunned by most researchers. The standard consensus among physicists has been that such problems are unstable, or hopelessly unpredictive [Wei]. This interpretation has also transferred into the mathematical community, and so, such problems are generally not considered.

Although it has been shown that the canonical 'initial value problem' for the wave equation with more than one time dimension is ill-posed in the sense of Hadamard [RR62, Wei], the recent work of S. Weinstein and W. Craig has presented a sufficient constraint on such problems endowing them with a well-posedness condition near-that of standard single time dimension problems [CW09]. This latter result will be briefly outlined in the conclusion and hopefully will be the subject of a later publication.

Both of the above arguments for the well-posedness of the single and multiple time wave equation are fleshed out in the proceeding sections, with specific care taken so that, with only limited knowledge of partial differential equations, they can be understood by a standard undergraduate audience.

2 The Wave Equation and The Theorem of Asgeirsson

Definition 2.1. In this paper, an *Initial Value Problem (IVP)* is a pair $[A, B]$ where A is a partial differential equation and B is a set of *initial conditions* which must be satisfied by a particular solution of A. Here, an *initial condition* is a set of function values imposed upon the solution at a specified point in its domain.

Definition 2.2. The *Laplacian* with respect to the vector $x = (x_1, x_2, \ldots, x_n)$, is the differential operator

$$\triangle_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to have *compact support* if $f = 0$ on all but a compact subset of $\mathbb{R}^n$. We will denote C^r functions of this variety by C_0^r.

Definition 2.4. A problem is said to be *well-posed (in the sense of Hadamard)* if each of the following hold:

- there exists a solution,
- the solution is unique,
- the solution depends continuously (in the chosen norm) on the data.

If any of the above conditions fail to hold, the problem is said to be *ill-posed*.

2.1 The wave equation in one time dimension

A useful and well-studied equation in modern mathematics, the 'wave equation' applied to a function $u = u(x, y, z, t)$, is defined as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

where we consider $x,\ y,\ z$ to be independent 'space variables' and t to be an independent 'time variable'. This equation is commonly used to describe the propagation of sound, light, and water waves as they move through their respective 3-dimensional medium.

When generalized to $(n+1)$-dimensional euclidean space, the wave equation is commonly seen in the form:

$$\frac{\partial^2 u}{\partial t^2} = u_{tt} = \triangle_x u, \tag{2.1}$$

where $n \geq 1$.

An interesting and relevant fact about Equation 2.1, tying into its physical significance, is that for every $n \geq 1$, the canonical IVP involving it is well posed on a salient class of physically valuable initial conditions. The usual case of this problem is given by the IVP $[(2.1), (2.2)]$.

Initial conditions: Take $f, g : \mathbb{R}^n \to \mathbb{R}$ where $n = 2k$ or $n = 2k+1$, and define

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{2.2}$$

The IVP $[(2.1, (2.2)]$ is commonly called a 'Cauchy Problem' for the wave equation and is well-posed provided that $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$ [RR62].

2.2 The wave equation in many time dimensions

Taking $t = (t_1, t_2, \ldots, t_m)$ and $x = (x_1, x_2, \ldots, x_n)$ the wave equation in many time dimensions is defined as

$$\triangle_t u = \triangle_x u.$$

Equivalently, as we will see useful later, when defining $y = (t_1, \ldots, t_{m-1})$ and redefining $t = t_m$, the above equation may be written

$$\frac{\partial^2 u}{\partial t^2} = (\triangle_x - \triangle_y)\, u. \tag{2.3}$$

As in section 2.1, we analogously construct corresponding initial conditions for Equation 2.3:

Initial conditions: Take $f, g : \mathbb{R}^n \times \mathbb{R}^{m-1} \to \mathbb{R}$ and define

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y). \tag{2.4}$$

We will soon see that $[(2.3), (2.4)]$ with $f \in C_0^k$ and $g \in C_0^k$ is ill-posed for every $k \geq 1$ (unless we have further restrictions upon f and g [CW09]). In order to prove this, we must first present an important theorem.

2.3 The theorem of Asgeirsson

Theorem 2.1 (Asgeirsson). For a solution $u \in C^2$ of the differential equation

$$\frac{\partial^2 u}{\partial t^2} = (\triangle_x - \triangle_y)\, u \quad (\text{where } m = n),$$

the average of $u(x, t_0)$ on a sphere of radius R and center x_0 in x-space is the same as the average of $u(x_0, t)$ on a sphere of radius R and center t_0 in t-space [Fri55].

Proof. For all $\alpha, \beta > 0$, define the ellipsoidal mean of u in $\mathbb{R}^n$

$$I(\alpha,\beta) = \frac{1}{2n\omega_{2n}} \int_{F(\alpha,\beta,x,t)=1} u(x,t)d\omega.$$

Where $F(\alpha,\beta,x,t) = \frac{|x|^2}{\alpha} + \frac{|t|^2}{\beta}$ and ω_k is the volume of the unit sphere in $\mathbb{R}^k$.

Now observe that because $d\omega$ is invariant under affine transformations, through a change of variables $x = \sqrt{\alpha}\eta$ and $t = \sqrt{\beta}\zeta$, we arrive with

$$I(\alpha,\beta) = \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2=1} u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)d\omega.$$

So

$$\begin{aligned}\frac{\partial}{\partial\alpha} I(\alpha,\beta) &= \frac{1}{2n\omega_{2n}\sqrt{\alpha}} \int\limits_{|\eta|^2+|\zeta|^2=1} \sum_{i=1}^{n} u_{x_i}(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)\eta_i d\omega \\ &= \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2=1} \nabla_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)\cdot\eta d\omega.\end{aligned}$$

Considering the vector $(\nabla_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta), 0_t)$, we get by the divergence theorem

$$\begin{aligned}\frac{\partial}{\partial\alpha} I(\alpha,\beta) &= \frac{1}{2n\omega_{2n}} \int\limits_{|\eta|^2+|\zeta|^2<1} \triangle_x u(\sqrt{\alpha}\eta, \sqrt{\beta}\zeta)d\eta d\zeta \\ &= \frac{1}{2n\omega_{2n}}(\alpha\beta)^{-\frac{n}{2}} \int\limits_{F(\alpha,\beta,y)<1} \triangle_x u(x,t)dxdt. \end{aligned} \tag{2.5}$$

Similarly, we arrive with

$$\frac{\partial}{\partial\beta} I(\alpha,\beta) = \frac{1}{2n\omega_{2n}}(\alpha\beta)^{-\frac{n}{2}} \int\limits_{F(\alpha,\beta,y)<1} \triangle_t u(x,t)dxdt. \tag{2.6}$$

So, we observe by Equations 2.5 and 2.6 that $(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta})I(\alpha,\beta) = 0$ since $(\triangle_x - \triangle_t)u = 0$. Thus, we have that $I(\alpha,\beta) = \phi(\alpha+\beta)$ for some $\phi : \mathbb{R} \to \mathbb{R}$, and in particular,

$$I(\alpha,\beta) = I(\beta,\alpha) \quad \text{for all } \alpha,\beta > 0. \tag{2.7}$$

Furthermore,

$$\begin{aligned}
\lim_{\beta\to 0^+} 2n\omega_{2n} I(\alpha,\beta) &= \int\limits_{|\eta|^2+|\zeta|^2=1} u(\sqrt{\alpha}\eta, 0)d\omega \\
&= \left[\frac{d}{dr}\int\limits_{|\eta|^2+|\zeta|^2\le r^2} u(\sqrt{\alpha}\eta,0)d\eta d\zeta\right]_{r=1} \\
&= \left[\frac{d}{dr}\int\limits_{|\eta|^2\le r^2} u(\sqrt{\alpha}\eta,0)\int\limits_{|\zeta|^2\le r^2-|\eta|^2} d\zeta d\eta\right]_{r=1} \\
&= \left[\frac{d}{dr}\omega_n\int\limits_{|\eta|^2\le r^2} u(\sqrt{\alpha}\eta,0)(r^2-|\eta|^2)^{\frac{n}{2}}d\eta\right]_{r=1} \\
&= n\omega_n\int\limits_{|\eta|^2\le 1} u(\sqrt{\alpha}\eta,0)(1-|\eta|^2)^{\frac{n-2}{2}}d\eta \\
&= n\omega_n\alpha^{1-n}\int\limits_{|x|^2\le\alpha} u(x,0)(\alpha-|x|^2)^{\frac{n-2}{2}}dx. \qquad (2.8)
\end{aligned}$$

Now, because the integrand in Equation 2.8 is bounded for all $|x|\le\alpha$, the above limit exists and we may extend the domain of I to include $\beta=0$. Similarly, we observe $\lim_{\alpha\to 0^+} I(\alpha,\beta)$ exists, and so we also extend I accordingly. We see now that by Equation 2.7, we have

$$I(\alpha,0) = I(0,\alpha) \quad \text{for all } \alpha>0. \qquad (2.9)$$

Finally, define the spherical means in x-space and t-space respectively, by

$$I_1(R) = \frac{1}{n\omega_n}\int\limits_{|x|=R} u(x,0)dS_x \quad\text{and}\quad I_2(R) = \frac{1}{n\omega_n}\int\limits_{|t|=R} u(0,t)dS_t. \qquad (2.10)$$

So when considering Equations 2.8 and 2.9 with $\alpha=R^2$, we find that for any fixed $R>0$

$$\begin{aligned}
0 &= I(\alpha,0)-I(0,\alpha) \\
&= \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int\limits_{|x|^2\le R^2} u(x,0)(R^2-|x|^2)^{\frac{n-2}{2}}dx \\
&\quad - \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int\limits_{|t|^2\le R^2} u(0,t)(R^2-|t|^2)^{\frac{n-2}{2}}dt \\
&= \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\int\limits_{|x|=r} u(x,0)dS_x dr \\
&\quad - \frac{\omega_n R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\int\limits_{|t|=r} u(0,t)dS_t dr \\
&= \frac{n\omega_n^2 R^{2-2n}}{2\omega_{2n}}\int_0^R r^{n-1}(R^2-r^2)^{\frac{n-2}{2}}\left(I_1(r)-I_2(r)\right)dr.
\end{aligned}$$

Thus, since $r^{n-1}(R^2 - r^2)^{\frac{n-2}{2}} > 0$ for all $0 < r < R$, it must be that $I_1(r) = I_2(r)$ for all $r > 0$ because $R > 0$ was considered arbitrarily.

Ultimately, although it has only been shown that the theorem holds for $x_0 = t_0 = 0$, because the differential equation is linear, the result holds under any translation of the points x_0 and t_0. □

Observe that the assumption $n = m$ in Asgeirsson's Theorem is artificial and that the result still holds if we are considering equations of the form $\triangle_x u = \triangle_t u$, where $n \neq m$. In such a case, we seek solutions independent of the neglected variables and the corresponding integration for the mean values is taken about x_0 and t_0 in $\max\{n, m\}$-dimensional space.

3 Solutions and Well-Posedness

3.1 One time dimension

Assuming $n = 2k$ or $n = 2k+1$, then because $u \in C^2$ it can be shown that for any $f \in C^{k+2}$ and $g \in C^{k+1}$ the following closed form solutions of $[(2.1), (2.2)]$ exist for all $n \geq 1$ [Sie10]. When $n = 1$ the solution is well-known, and given by D'Alembert's formula

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

Furthermore, defining $c_n = 1 \cdot 3 \cdot \dots \cdot (2k-1)$, we have for all other odd values of n

$$\begin{aligned} u(x,t) = {} & \frac{1}{c_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int\limits_{|\xi|=1} f(x+t\xi)dS_\xi\right) \\ & + \frac{1}{c_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int\limits_{|\xi|=1} g(x+t\xi)dS_\xi \end{aligned}$$

Alternatively for all even n, using the method of descent upon the odd solutions above, we arrive with

$$\begin{aligned} u(x,t) = {} & \frac{1}{c_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-2}\int\limits_{|\xi|\leq 1} \frac{f(x+t\xi)}{\sqrt{1+|\xi|^2}}d\xi\right) \\ & + \frac{1}{c_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} t^{n-2}\int\limits_{|\xi|\leq 1} \frac{g(x+t\xi)}{\sqrt{1+|\xi|^2}}d\xi. \end{aligned}$$

We see by above formulas that the domain of dependence of the solution at (x,t) for $t > 0$ is $\{x + t\xi : |\xi| \leq 1\}$. This allows us to conclude that because the functions f and g in Equation 2.2 have compact support, so must the solution u when the t-variable is considered fixed. In particular, for any $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$ there is some $R > 0$ such that

$$u(x,t) = 0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_{R+t}(0). \tag{3.1}$$

Accounting for the above considerations, it will now be shown that all solutions to $[(2.1), (2.2)]$ with $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$, are unique. In essence, this suffices to show that for all $f \in C_0^{k+2}$ and $g \in C_0^{k+1}$, the IVP $[(2.1), (2.2)]$ is well-posed because the continuous dependence of $[(2.1), (2.2)]$ follows directly from the closed form of the solutions above.

Proof of Uniqueness: Let u_1 and u_2 both be solutions of $[(2.1),(2.2)]$. Then for $v = u_1 - u_2$, we have that since the wave equation is linear,

$$\begin{cases} v_{tt} = \triangle_x v & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = f(x) - f(x) = 0 & \text{for } x \in \mathbb{R}^n \\ v_t(x,0) = g(x) - g(x) = 0 & \text{for } x \in \mathbb{R}^n \end{cases} \tag{3.2}$$

i.e. v satisfies $[(2.1),(2.2)]$ for $f = g = 0$.

Consider now, the global energy function arising from physical considerations,

$$E(t) = \int_{\mathbb{R}^n} \left({v_t}^2 + |\nabla_x v|^2 \right) dx,$$

and observe that

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^n} \left(v_t v_{tt} + \sum_{i=1}^n v_{x_i} v_{x_i t} \right) dx \\ &= \int_{B_{R+t}(0)} \left(v_t v_{tt} + \nabla_x v \cdot \nabla_x v_t \right) dx \\ &= \int_{B_{R+t}(0)} \left(v_t v_{tt} - v_t \triangle_x v \right) dx + \int_{\partial B_{R+t}(0)} v_t \nabla_x v \cdot \hat{\nu} dS \\ &= \int_{B_{R+t}(0)} v_t \cdot 0 dx + 0 \\ &= 0, \end{aligned}$$

where the second and fourth equality follow from Equations 3.1 and 3.2, and the third follows from Green's first identity.

Thus, for all $t > 0$, $E(t) = E(0) = 0$ and so

$$\nabla_x v = 0 \quad \text{and} \quad v_t = 0 \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Thus v is constant, which by Equation 3.2 implies $v = 0$ and so the uniqueness condition is indeed satisfied. □

3.2 Many time dimensions

Courant and Hilbert's classic argument will now be constructed in order to show that $[(2.3),(2.4)]$ is ill-posed [CH62].

The problem of determining functions from their mean values: Consider the spherical mean for a radius r of a function $u = u(y,t)$ centered at $(y,0)$ in (y,t)-space:

$$M_u(y,r) = \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2=r^2} u(y+\xi,\tau) dS = Q[u].$$

Observe that through the symmetry of the $t=0$ coordinate, $Q[u]$ depends only upon the even part of u in the t-variable, mainly $\frac{1}{2}(u(y,t)+u(y,-t))$. We seek to determine $u(y,t)+u(y,-t)$ from a particular known $M_u(y,r)$, and so through incredible foresight, we define

$$N_u(y,r) = \int_0^r M_u(y,\rho) d\rho = \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2 \le r^2} u(y+\xi,\tau) d\xi \; d\tau. \tag{3.3}$$

Although it may appear unmotivated at the moment, differentiating N_u with respect to any of the y_i variables gives

$$\begin{aligned}\frac{\partial}{\partial y_i} N_u(y,r) &= \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2\le r^2} u_{y_i}(y+\xi,\tau)\ d\xi\ d\tau \\ &= \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)\hat{\nu_i}\ dS \\ &= \frac{1}{n\omega_n r} \int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)\xi_i\ dS,\end{aligned}$$

where the second and third equalities follow by the divergence theorem and because $\hat{\nu_i} = \frac{\xi_i}{r}$. Thus, observe that

$$\begin{aligned}Q[u(y,t)y_i] &= \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2= r^2} u(y+\xi,\tau)(y_i+\xi_i)dS \\ &= y_i M_u(y,r) + r\frac{\partial}{\partial y_i} N_u(y,r) \\ &= y_i M_u(y,r) + r\frac{\partial}{\partial y_i}\int_0^r M_u(y,\rho)d\rho \\ &= D_i M_u,\end{aligned}$$

where

$$D_i = (y_i + r\frac{\partial}{\partial y_i}\int_0^r \cdot\ d\rho)$$

is a linear operator on the functions $M_u(y,r)$.

Through linearity, we now see that given a polynomial $P : \mathbb{R}^n \to \mathbb{R}$, we have

$$Q[Pu] = P(D_1, \ldots, D_n)M_u$$

and so $M_{Pu}(y,r) = Q[Pu]$ will be known given that we have M_u.

Alternatively, we have

$$\begin{aligned}Q[Pu] &= \frac{1}{n\omega_n} \int\limits_{|\xi|^2+\tau^2= r^2} P(y+\xi)u(y+\xi,\tau)dS_{\xi,\tau} \\ &= \frac{1}{n\omega_n} \int\limits_{|y-\eta|^2+\tau^2= r^2} P(\eta)u(\eta,\tau)dS_{\eta,\tau} \\ &= \frac{1}{2n\omega_n} \int\limits_{|y-\eta|^2+\tau^2= r^2} P(\eta)(u(\eta,\tau)+u(\eta,-\tau))dS_{\eta,\tau},\end{aligned}$$

where in the above we have taken $\eta = y+\xi$ and considered that $Q[u]$, and hence $Q[Pu]$, depend only upon the even part of u in the t-variable.

Now, for $|y-\eta|^2+\tau^2 = r^2$, we consider $\tau \ge 0$. So, writing $\tau = \phi(\eta) = \sqrt{r^2 - |\eta-y|^2}$ we get

$$\begin{aligned}dS_{\eta,\tau} &= \sqrt{1+|\nabla_\eta \phi|^2}d\eta \\ &= \frac{\sqrt{\tau^2+|\eta-y|^2}}{\tau}d\eta \\ &= \frac{r}{\tau}d\eta.\end{aligned}$$

Thus,

$$Q[Pu] = \frac{r}{2n\omega_n} \int\limits_{|y-\eta|^2 \le r^2} P(\eta)(u(\eta,\tau) + u(\eta,-\tau))\frac{d\eta}{\tau}. \tag{3.4}$$

Finally through the Stone-Weierstrass Theorem, because polynomials are dense in $C(\overline{B}_r(0), \mathbb{R})$ under the supremum norm, we have that the function $\frac{1}{\tau}(u(\eta,\tau) + u(\eta,-\tau))$ can be determined uniquely by $Q[Pu] = P(D_1, \ldots, D_n)M_u$. We then receive from this, a unique determination of the even part of $u(y,t)$ for $|y_0 - y|^2 + t^2 = r^2$.[1] The details of this unique determination result distract from the proof, see Theorem A.1 in the appendix.

Claim 3.1. Given any sufficiently small $\epsilon > 0$, any $y_0 \in \mathbb{R}^n$, and any $r_0 > 0$, the even part of u, $\frac{1}{2}(u(y,t) + u(y,-t))$, upon the sphere $|y_0 - y|^2 + t^2 \le r_0^2$ is determined uniquely by $M_u(y_0, r_0)$ upon the finite cylinder, $0 \le r < r_0$ and $|y - y_0| \le \epsilon$.

Proof. Observe that in order to calculate $D_i M_u = y_i M_u(y,r) + r\frac{\partial}{\partial y_i}\int_0^r M_u(y,\rho)d\rho$ for y_0 and r_0 we need only to know $M_u(y,r)$ in some neighbourhood of y_0 in y-space and for $0 \le r < r_0$. Without loss of generality, assume that this necessary neighbourhood is a ball of radius $\epsilon > 0$ centered at y_0. We now have that in order to calculate $Q[Pu]$, it is necessary only to know $M_u(y,r)$ for $0 \le r < r_0$ and $|y - y_0| \le \epsilon$.

Furthermore, recalling Equation 3.4 and noting that we are only considering $0 \le r < r_0$, we find that M_u in the above cylinder uniquely determines the even part of u in the entire solid sphere $|y_0 - y|^2 + t^2 \le r_0^2$. □

ILL-POSEDNESS: Before we continue, it is relevant to note that any solution of $[(2.3), (2.4)]$ will be even in the t-variable because the map $t \mapsto -t$ preserves (2.3).

Now recall $[(2.3), (2.4)]$, i.e.

$$\begin{cases} u_{tt} = (\triangle_x - \triangle_y)\, u & \text{for } (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^m \\ u(x,y,0) = f(x,y) & \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \\ u_t(x,y,0) = g(x,y) & \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \end{cases}$$

and let G be a domain $G \subset \mathbb{R}^n$, $\epsilon > 0$, and consider only $y \in B_\epsilon(y_0)$ and $x \in G$.

Consider a solution u to $[(2.3), (2.2)]$. Then for fixed x we note that our prescribed function f determines M_u over all spheres in (y,t)-space such that $(y,t) \in B_\epsilon(y_0) \times \{0\}$ and whose radius r_0 is still small enough so that $B_{r_0}(x) \subset G$.

Recalling the proof of Asgeirsson's Theorem we now arrive with the following two cases:

CASE 1 $(m \ge n)$: By Asgeirsson's Theorem directly, we find

$$\frac{1}{m\omega_m} \int\limits_{|\zeta'|^2 = r_0^2} u(x+\zeta, y, 0)dS_{\zeta'} = \frac{1}{m\omega_m} \int\limits_{|\xi|^2+\tau^2 = r_0^2} u(x, y+\xi, \tau)dS_{\xi,\tau},$$

where and $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\zeta' = (\zeta_1, \ldots, \zeta_n, \ldots, \zeta_m)$.

[1] Although the function $\frac{1}{\tau}(u(\eta,\tau) + u(\eta,-\tau))$ is not continuous when $\tau = 0$ (i.e. when $|\eta - y| = r$), it is continuous on any compact set inside $B_r(0)$, and so essentially implying our conclusion [CH62].

Case 2 ($n \geq m$): Invoking Asgeirsson's Theorem again, we find

$$\begin{aligned}
&\frac{1}{n\omega_n} \int\limits_{|\zeta|^2=r_0^2} u(x+\zeta, y, 0)dS_\zeta \\
&= \frac{1}{n\omega_n} \int\limits_{|\xi'|^2+\tau^2=r_0^2} u(x, y+\xi, \tau)dS_{\xi',\tau} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr} \int\limits_{|\xi'|^2+\tau^2\leq r^2} u(x, y+\xi, \tau)d\xi' d\tau \right]_{r=r_0} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr} \int\limits_{|\xi|^2+\tau^2\leq r^2} u(x, y+\xi, \tau) \int\limits_{|\xi'-\xi|^2\leq r^2-|\xi|^2-\tau^2} d\xi' d\tau \right]_{r=r_0} \\
&= \frac{1}{n\omega_n} \left[\frac{d}{dr}\omega_{n-m} \int\limits_{|\xi|^2+\tau^2\leq r^2} u(x, y+\xi, \tau)(r^2-|\xi|^2-\tau^2)^{\frac{n-m}{2}} d\xi d\tau \right]_{r=r_0} \\
&= \frac{(n-m)\omega_{n-m} r_0}{n\omega_n} \int\limits_{|\xi|^2+\tau^2\leq r_0^2} u(x, y+\xi, \tau)(r_0^2-|\xi|^2-\tau^2)^{\frac{n-m-2}{2}} d\xi d\tau,
\end{aligned}$$

where $\xi = (\xi_1, \ldots, \xi_{m-1})$ and $\xi' = (\xi_1, \ldots, \xi_{m-1}, \ldots, \xi_{n-1})$. So in this case, similar to the proof of Asgeirsson's Theorem, we find that for all $0 \leq r < r_0$, the above equation may be rewritten:

$$\frac{1}{n\omega_n} \int\limits_{|\zeta|^2=r^2} u(x+\zeta, y, 0)dS_\zeta = \frac{m(n-m)\omega_{n-m}\omega_m r}{n\omega_n} \int_0^r \rho^{n-m-1}(r^2-\rho^2)^{\frac{n-m-2}{2}} I(\rho)d\rho, \qquad (3.5)$$

where

$$I(r) = \frac{1}{m\omega_m} \int\limits_{|\xi|^2+\tau^2=r^2} u(x, y+\xi, \tau)dS_{\xi,\tau}$$

is the spherical mean of u for a radius r, M_u, for a fixed $x \in G$ taken in (y,t)-space at the point $(x, y, 0)$. Now, because the left hand side of Equation 3.5 is determined by f, differentiating Equation 3.5 twice with respect to r gives us a first order differential equation for $I(r)$ which has a unique solution by the existence and uniqueness theorem for ordinary differential equations.

Given arbitrary n and m, it is necessary that we restrict $r_0 \geq 0$ to be small enough in order that $B_{r_0}(x) \subset G$, i.e. so that the integral

$$\int\limits_{|\zeta|^2=r_0^2} u(x+\zeta, y, 0)dS_\zeta$$

is well defined. With this considered, we have by our previous claim that the even function $\frac{1}{2}(u(x,y,t) + u(x,y,-t))$, and thus also u itself, is uniquely determined in the sphere $|y_0-y|^2 + t^2 \leq r_0^2$ by it's mean value M_u in (y,t)-space such that $(y,t) \in B_\epsilon(y_0) \times \{0\}$. Furthermore, consider that M_u itself is equal to the same integral of the prescribed function $f(x,y)$, and so M_u is determined uniquely by $f(x,y)$.

In an analogous way, the function $\frac{1}{2}(u_t(x,y,t) + u_t(x,y,-t))$ is determined uniquely by $g(x,y)$ and from this we get that $u(x,t)$ is determined uniquely by f and g. In particular, we get that $u(x,y,0)$ is determined for its initial value $t = 0$ inside the sphere in y-space, $|y_0-y|^2 \leq r_0^2$.

Thus, we have proven that *if the initial values of a solution u of* [(2.3), (2.4)] *are known for $x \in G$ and t in an arbitrarily small sphere $|y_0-y|^2 \leq \epsilon^2$, then the initial values are uniquely determined everywhere in the larger sphere $|y_0-y|^2 \leq r_0^2$, where r_0 is defined above* [CH62].

Hence, arbitrary initial conditions cannot be imposed upon the wave equation with multiple time dimensions. This is in violation of any general well-posedness condition analogous to $[(2.1),(2.2)]$ since the existence of solutions will fail if the initial conditions f and g are not properly prescribed.

4 Conclusion

A fundamental difference between the two initial value problems $[(2.1),(2.2)]$ and $[(2.3),(2.4)]$ has now been highlighted. This being that the first problem is well-posed, and so in loose terms, physically significant; while the second problem is ill-posed and so not likely physically significant.

A possible physical interpretation of this is due specifically to the fact that when considering $[(2.3),(2.4)]$, we are prescribing our initial conditions on a 'mixed hypersurface'—this being a hypersurface extending not only in space, but also in time [Wei]. Thus the characteristics where upon a possible solution would propagate are time-like in some directions and must agree with the time-like prescribed initial conditions. This would inhibit a certain knowledge of future conditions which is highly unphysical.

In the past two years notable advancements of the well-posedness in the wave equation with many times have been pursued by W. Craig and S. Weinstein [CW09]. Their methods outline a specific Fourier transform derived constraint which, when imposed upon the respective initial conditions, has illuminated a new set of conditions bringing well-posedness to the initial value problem $[(2.3),(2.4)]$. An addendum article is planned to explore this particular development.

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A Unique determination of functions from dense integral operators

Theorem A.1. Let $r > 0$ and assume $f \in C(\overline{B}_r(0), \mathbb{R})$. Then, if

$$\int_{|x|\leq r} P(x)f(x)dx$$

is known for all $P \in S \subset C(\overline{B}_r(0), \mathbb{R})$, where S is some subset dense in supremum norm, we have a unique determination of the function f.

Proof. Suppose we have two functions, $f_1, f_2 \in C(\overline{B}_r(0), \mathbb{R})$ which have the same values under the operation $\int\limits_{|x|<r} P(x) \cdot dx$ for any $P \in S$. Then

$$\int\limits_{|x|\leq r} P(x)(f_1 - f_2)dx = 0$$

for all $P \in S$. Effectively, in order to show $f_1 = f_2$, we only need to show that if $f \in C(\overline{B}_r(0), \mathbb{R})$ and for all $P \in S$

$$\int\limits_{|x|\leq r} P(x)f(x)dx = 0,$$

then f must be the zero function.

So suppose that this is the case, but that $f \neq 0$. Then, since f is continuous, f is measurable, so $\operatorname{sgn} f \in L^1(B_r(0))$, so by the density of $C(\mathbb{R}^n, \mathbb{R})$ in L^1 we can find a sequence from S, $P_n(x)_{n=1}^{\infty}$, converging uniformly to $\operatorname{sgn} f$. Then let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|P_n - \operatorname{sgn} f\|_\infty < \frac{\epsilon}{\|f\|_1}.$$

We see that

$$\begin{aligned}
\|f\|_1 &= \left| \int\limits_{|x|<r} |f|(x)dx \right| \\
&= \left| \int\limits_{|x|<r} (f(x) \cdot \operatorname{sgn} f - P_n(x)f(x))dx \right| \\
&\leq \int\limits_{|x|<r} |f(x) \cdot (\operatorname{sgn} f - P_n(x))|dx \\
&\leq \|f\|_1 \|\operatorname{sgn} f - P_n\|_\infty \\
&< \epsilon
\end{aligned}$$

by Holders inequality. Since ϵ is arbitrary this is a contradiction, so $f = 0$. □