Relativistic Fluid Dynamics

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Abstract: Understanding the evolution of a many bodied system is still a very important problem in modern physics. Fluid mechanics provides a mechanism to determine the macroscopic motion of the system. These equations are additionally complicated when we consider a fluid moving in a curved spacetime. The following paper discusses the derivation of the relativistic equations of motion, uses numerical methods to provide solutions to these equations and describes how the curvature of spacetime is modified by the fluid.

1 Introduction

Traditionally, a fluid is defined as a substance that does not support a shear stress. This definition is somewhat lacking, but it does present the idea that fluids "flow" and distort. Any non-rigid multi-bodied state can, under a suitable continuum hypothesis, be thus described as a fluid and will follow certain equations of motion. Here, we define a relativistic fluid as classical fluid modified by the laws of special relativity and/or curved spacetime (general relativity). The following paper attempts to provide a basic introduction to these equations of motion of a relativistic fluid.

Fluid dynamics is an approximation of the motion of a many body system. A true description of the evolution of a fluid would, in principle, need to account for the motion of each individual particle. However, this description is impractical and of no substantial worth when modelling sufficiently large systems. Therefore, provided that the desired level of accuracy is much lower than the continuum approximation, it is acceptable to consider a system as a fluid. The applications of such an approximation to relativistic fluids are varied and have been applied to the many different domains from plasma physics to astrophysics.

In this discussion, we begin with introducing the relevant equations found in Newtonian fluid mechanics. We follow this with an introduction to the necessary mathematics to describe a four dimensional curved spacetime. The stress-energy tensor of a perfect fluid is introduced and the equations of motion of a relativistic fluid are derived. We briefly mention the modification of the stress-energy tensor in the presence of viscosity. We finish off with a simple calculation of how the stress-energy of the fluid in question modifies the curvature of space-time. The reader is assumed here to have a basic understanding of relativity along with a low-level understanding of Newtonian fluid dynamics.

We note that for the remainder of this paper with will use units such that $c = G = 1$.

2 Introductory Mathematics

Classical fluids have, from a theoretical perspective, played a very important role in developing a great deal of the mathematics of vector calculus of partial differential equations (PDEs), and forms the core of our understanding of problems in multi-body physics. Extension of this classical field to the domain of relativity requires the use of the understanding of motion in a curved spacetime. An introduction to the mathematics required for this development is provided here.

2.1 Classical fluids

Determining solutions to the classical equations of motion of a fluid is still a very active area of research. If we just consider a Newtonian fluid (water and air are both good examples of this type), the strain tensor can be written as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right),$$

where u^j, x^j are the j^{th} components of the fluid velocity and coordinate vectors. For the remainder of this paper, we will employed the Einstein summation convection $x^j x_j = \sum_j x^j x_j$ over the range of the indices.

In this domain, there are still four basic equations to satisfy

1. *Continuity equation.* This equation is derived through the hypothesis of conservation of mass. In its typical form, it can be written down as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0,$$

   where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x^j}$ called the material derivative, and ρ the density of the fluid.

2. *Momentum equation.* The fluid must also conserve momentum. We ensure this by requiring that

$$\frac{Du_j}{Dt} = \frac{\partial T_{ij}}{\partial x_j} - \rho g_j$$
$$T_{ij} = -P\delta_{ij} + 2\mu e_{ij} + \lambda e_{mm}\delta_{ij},$$

   where T_{ij} is the stress tensor, P is the pressure , g_j is the constant gravitational acceleration, δ_{ij} is the unity matrix, with μ and λ are fluid dependent scalars. If $\mathbf{u}$ is incompressible, ($\nabla\cdot\mathbf{u} = 0$) these equations reduce to the Navier-Stokes Equations.

3. *Equation of state.* This equation defines the relation between pressure (P), temperature (T), and density (ρ). This equation can vary depending on the fluid in question. For an ideal gas it can be written

$$P = \rho RT$$

   with constant R.

4. *Temperature/Energy equation.* This final equation is needed to deal with the thermodynamic effects within the medium. If we consider the heat flux vector q_i at any given point, we need to solve for internal energy e

$$\rho\frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} - P\left(\frac{\partial u_i}{\partial x_i}\right) + \phi$$

   with density (ρ), pressure (P), velocity (u_i), and viscous dissipation (ϕ). This equation indicates that the change in energy is due to convergence of heat, volume compression and viscous dissipation.

All this gives us a system of six, non-linear coupled PDEs. These equations have been included to help guide the reader in understanding how the following equations reduce in the Newtonian limit. It is important to realize that these equations have still not been solved and currently represent one of the most challenging problems in applied mathematics. For further details, Kundu [Kun90] has a well written text on classical fluid mechanics.

2.2 Curved spacetime

In order to understand relativistic fluids, if becomes important to develop the mathematical tools to look at curves in a curved spacetime. While the reader is assumed to have a basic knowledge of differential geometry, a brief outline of the some of the mathematics is presented here.

The length of an infinitesimally small line element in 4-space can be found by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where μ,ν run from $\{0, 1, 2, 3\}$ or, equivalently, $\{t, x, y, z\}$ in Cartesian spacetime. Note that this line element is invariant of the chosen coordinate system, that is, it is a scalar. Here, the metric $g_{\mu\nu}$ (of form - + + +) serves the role of a weighting function, used in defining the length of a path. In a curved space the placement of the index is very important, and we use the metric $g_{\mu\nu}$ to raise and lower the indices.

$$V^\mu = g^{\mu\nu} V_\nu \qquad \text{or} \qquad V_\mu = g_{\mu\nu} V^\nu.$$

Consequently,

$$V^\mu = g^{\mu\nu} g_{\nu\rho} V^\rho \qquad \to \qquad g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho, \tag{2.1}$$

where δ^μ_ρ is the Kronecker delta.

The infinitesimal length of a curve ds^2 divides up into three different regimes.

1. *Timelike.* If $ds^2 < 0$, the curve is called timelike. Two events are timelike separate if there exists some rest frame, in which both events occur at the same location at different times.
2. *Null.* If $ds^2 = 0$, the curve is null. There does not exist a rest frame.
3. *Spacelike.* If $ds^2 > 0$, the curve is spacelike. Two events are spacelike separated if there exists some rest frame, in which both events occur at the same time at different locations.

It turns out that all matter travels along timelike curves and light moves along null paths. As such, it is possible to define, for timelike curves, a proper time (τ) which is the time measured by an observer in a rest frame.

$$d\tau^2 = -ds^2. \tag{2.2}$$

Using this definition, it is possible to define the 4-vector velocity

$$u^\mu = \frac{dx^\mu}{d\tau}.$$

As a quick aside, the purpose of introducing this tensor calculus is to allow for a derivation of physical laws, independent of a particular coordinate system. As such, a tensor will necessarily obey certain transformation laws. We provide here the transformation relation for a vector, with higher order tensors transforming in a consistent manner.

$$\bar{V}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} V_\nu \qquad \text{or} \qquad \bar{V}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} V^\nu.$$

2.2.1 The covariant derivative and the material derivative

It is important to know how to find the derivative at a given point of a vector field. In a flat spacetime, the rate of change of some vector field V^ν in a particular direction x^μ can be found simply by taking the partial derivative. However, the derivative is not so easy to define in a curved spacetime. As an example, consider the vector $\mathbf{V} = V^\mu \mathbf{e}_\mu$ where $\mathbf{e}_\mu$ is some basis vector at a point. In a flat Cartesian coordinate system, the basis vectors are constant, but in a curved spacetime, they are not. We see then that

$$\partial_\mu \left(V^\nu \mathbf{e}_\nu\right) = \left(\partial_\mu V^\nu\right) \mathbf{e}_\nu \qquad \text{(Flat Cartesian)}$$
$$\partial_\mu \left(V^\nu \mathbf{e}_\nu\right) = \left(\partial_\mu V^\nu\right) \mathbf{e}_\nu + V^\nu \partial_\mu \mathbf{e}_\nu. \qquad \text{(Curved Space)}$$

From this example it has been shown that there are two main issues to resolve when defining the derivative in a curved space. First, how does one find a limit in a curved spacetime? And second, how do we ensure that the derivative transforms correctly. It can be shown that both of these requirements can be met by defining the covariant differential operator

$$\begin{aligned} \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma \\ \Gamma_{\mu\nu\rho} &= \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho}\right), \end{aligned} \tag{2.3}$$

where commas denote partial derivatives. Here we see that the connection coefficient, Γ, "corrects" for the curvature of the space. Similarly, for a rank-2 tensor, we can write

$$\nabla_\rho T^{\mu\nu} = \partial_\rho T^{\mu\nu} + \Gamma^\mu_{\sigma\rho} T^{\sigma\nu} + \Gamma^\nu_{\sigma\rho} T^{\mu\sigma}.$$

Before we continue, we quickly write down a few important identities which will be important later. First, the material derivative can be written,

$$\frac{D}{D\tau} V^\nu(x_\mu(\tau)) = \frac{\partial x^\mu}{\partial \tau} \nabla_\mu V^\nu = V^\mu \nabla_\mu V^\nu.$$

Secondly, it can also be shown that for timelike curves, by Equation 2.2, that

$$u^\mu u_\mu = -1 \qquad \Rightarrow \qquad u_\mu \nabla_\nu u^\mu = 0. \tag{2.4}$$

This identity will prove invaluable when working through the details below. Finally, we note that for Riemann manifold (considered here),

$$\nabla_\mu g_{\mu\nu} = 0 \tag{2.5}$$

as a result of the definition of the connection coefficients.

2.2.2 Curvature and the Riemann tensor

We briefly present the ideas here simply for completeness and the details of the following calculations have been omitted. This section is presented merely to remind the reader of where the Einstein field equations have their origins. Anderson [AC07] has a good discussion of many these concepts.

The measure of the curvature of space is defined in terms Riemann tensor ($R^\mu_{\nu\rho\sigma}$), Ricci tensor ($R_{\mu\nu}$), and Ricci scalar (R).

$$\begin{aligned} R^\mu_{\nu\rho\sigma} &= \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\tau\rho}\Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma}\Gamma^\tau_{\nu\rho} \\ R_{\mu\nu} &= R^\rho_{\mu\rho\nu} \\ R &= R^\rho_\rho \end{aligned} \tag{2.6}$$

From the Bianchi identities

$$\nabla_\lambda R^\mu_{\nu rho\sigma} + \nabla_\rho R^\mu_{\nu\sigma\lambda} + \nabla_\sigma R^\mu_{\nu\lambda\rho} = 0,$$

it can be shown that

$$\nabla_\nu \left[R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right] = 0.$$

As such, we define the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$

2.3 Einstein field equations

John Wheeler once said

> "Mass tells space-time how to curve, and space-time tells mass how to move."

The Einstein tensor is a measure of the curvature of spacetime. Mass is merely a form of energy and, as such, we denote the stress-energy tensor, $T_{\mu\nu}$, containing all of the information of the energy of a system. Thus, these two tensors must be in balance, which is represented in the Einstein field equations (EFE)

$$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}, \tag{2.7}$$

where we include the constants c, G to present the EFE in their usual form. Recall that we are using units such that $c = G = 1$.

The EFE represent a system of ten non-linear partial differential equations. The complexity of these equations explains why few analytical solutions exist.

We've seen above that

$$\nabla_\nu G^{\mu\nu} = 0$$

applying this to Equation 2.7

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.8}$$

This equation is very important in fluid dynamics, as we shall see. This equation encapsulates the idea of energy and momentum conservation.

3 Governing Equations

One of the most difficult aspects of relativistic fluid dynamics is keeping track of "what-goes-where", and what index corresponds to what physical property. In Newtonian fluids, all of the equations clearly have their own distinct physical interpretation, but when we extend these ideas to higher dimensions it is important keep track of what physics we are referring too.

It may not appear clear, however, how Equation 2.8 relates to the standard Newtonian fluid dynamics described above. The easiest way to compare these two is to first define projection operators, which will allow us to understand this equation from a more intuitive front. Anile [Ani89] has a good description of introductory relativistic fluid mechanics and the use of these projectors.

3.1 Projections

For any timelike curve p_μ, we can project this into its timelike and spacelike components. To project it into its pure timelike contribution, we contract p_μ onto u^μ. This projection captures what occurs in the rest frame of an observer as he travels along with the fluid. This is sometimes associated with "Lagrangian" coordinates.

Alternatively, sometimes it is valuable to project an equation into its purely spacelike components. We do this by defining

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \qquad \text{or} \qquad h^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu .$$

It is left to the reader to observe that the timelike projection u^μ and the spacelike projection h^μ_ν are orthogonal.

We then find that Equation 2.8 can be decomposed into an energy conservation component

$$u_\nu \nabla_\mu T^{\mu\nu} = 0$$

and a momentum conservation component

$$h_{\rho\nu} \nabla_\mu T^{\mu\nu} = 0.$$

3.2 Stress energy tensor

Different systems will have different stress energy tensors. Often, a lot of the problems of viscosity and other effects can be neglected compared with pressure or other more dominant effects. We will consider here the "perfect fluid" stress energy tensor which is the one typically introduced when approaching the subject for the first time.

In the rest frame of the observer it can be written

$$T^{\mu\nu} = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}_{\mu\nu} ,$$

or, in a more general frame,

$$T^{\mu\nu} = (e + p)\, u^\mu u^\nu + p g^{\mu\nu}, \tag{3.1}$$

with $g^{\mu\nu}$ the metric, p pressure, e the total energy density.

Typically, we can write out that

$$e = \rho(1 + \epsilon),$$

with ρ the rest frame mass energy density and ϵ internal energy density per unit mass.

The continuity equation can be written down as the following

$$\nabla_\mu (\rho u^\mu) = 0,$$

which ensures conservation of mass.

A more general conservation energy equation of this system can then be derived by finding the timelike component of Equation 2.8, projecting it onto u_ν:

$$u^\mu \nabla_\mu e = -\,(e + p)\, \nabla_\mu u^\mu, \tag{3.2}$$

where we recall that $\nabla_\mu g^{\mu\nu} = 0$, in the space we are considering here, and we have used the identities of Equation 2.4.

Similarly, we can project Equation 2.8 into its spacelike component using $h_{\alpha\mu}$:

$$(e+p)u^\mu \nabla_\mu u^\alpha = -h^{\alpha\mu}\nabla_\mu p, \tag{3.3}$$

where, again, we have used the identities found in Equation 2.4

What we have shown here are the relativistic equivalent equations to the momentum and mass conservations equations given in the Newtonian regime. In a relativistic case, it is the conservation of energy, not mass, which concerns us. However, as we return to the Newtonian domain, other sources of energy (kinetic, etc.) tend to be dominated by mass.

The equation of state and the conservation of temperature equations are not so easy to find. These need to be derived statistically using thermodynamic principles for the fluid in question.

3.3 Relativistic Euler equations

Our goal is to write out a system of equations which can be used to solve for the flow of a fluid. At this point we have a conservation of energy equation (Equation 3.2) and a conservation of momentum equation (Equation 3.3). It is insightful to compare these equations with their classical counterparts in order to help understand what these equations mean. We will do this by expanding Equation 3.3

$$(e+p)u^\mu \nabla_\mu u_\nu = -\nabla_\nu p - u_\nu u^\mu \nabla_\mu p,$$

from which we can write out the spatial components as

$$(e+p)\frac{D\mathbf{u}}{D\tau} = -\nabla p - \mathbf{u}\frac{Dp}{D\tau} \qquad \text{(Momentum equation)}$$

$$u^\mu \nabla_\mu e = -\left(e+p\right)\nabla_\mu u^\mu, \qquad \text{(Continuity equation)}$$

where $\frac{D}{D\tau} = u^\mu \nabla_\mu$. Now we see that if in the low velocity limit ($u_i \ll 1$), with $e \gg p$, $e \approx \rho$, and the fluid is incompressible ($\nabla \cdot \mathbf{u} = 0$) as is typical with water, we get back out typical Euler equations of Newtonian fluids. (Incompressible, viscous free Navier-Stokes equations.)

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p$$

$$\frac{D\rho}{Dt} = 0.$$

For a more detailed look at Newtonian fluids, see Kundu [Kun90].

We stop here and see that we have extended the Newtonian fluid equations into their relativistic form. Of course we are missing three very important items from this derivation. We have left out all viscosity terms, temperature evolution, and we still have not yet written down an equation of state. These are three fundamental properties which we have ignored here. The reason for this is simple; these additions are very complicated. We shall discuss these further in this article, however, they have been omitted here in order to aid the reader in understanding the current physical content.

We should also note here that we have assumed a known metric for our purposes. This is often acceptable for certain application; however, a more general relativistic treatment is required when the fluid itself causes space-time to curve.

3.4 Viscosity

Before continuing, we note that the we have omitted from the equations of motion viscosity. The addition of viscosity to a relativistic fluid will amount to an addition of non-diagonal terms to the stress energy tensor. These terms drastically increase the complexity of the equations. Viscosity plays an important role in the dispersion of energy of a system and are indispensable in the study of turbulence.

We present here the modification of the stress-energy tensor as a result of viscosity. We compare these terms to their classical counterpart. Landau [LL59] has a brief discussion on the topic (Alternatively, see the book by Wilson and Mathews [WM03]).

$$
\begin{aligned}
T_{\mu\nu} &= pg_{\mu\nu} + (e+p)u_\mu u_\nu + \tau_{\mu\nu} \\
\tau_{\mu\nu} &= -\eta\left(\left[\underbrace{\nabla_\mu u^\nu + \nabla_\nu u^\mu}\right] + u_\mu u^\alpha \nabla_\alpha u_\nu + u_\nu u^\alpha \nabla_\alpha u_\mu\right) - (\zeta - \frac{2}{3}\eta)\left[\underbrace{\nabla_\alpha u^\alpha}\right](g_{\mu\nu} + u_\mu u_\nu)
\end{aligned}
$$

Here we emphasize the relation to the Newtonian case. η and ζ are coefficients of viscosity.

4 Steady State Solution

The simplest relativistic fluid derivation is the hydrostatic problem. In the case, we can assume that the fluid is at rest and we write out that

$$u_0 = \sqrt{-g_{00}} \qquad u_i = 0,$$

which is simply stating that the fluid has no velocity in 3-space.

Looking back to the momentum equation (Equation 3.3) and Equation 2.3, we find that

$$
\begin{aligned}
-(e+p)\Gamma^0_{\nu 0} u_0 u^0 &= -\nabla_\nu p \\
\frac{1}{e+p}\nabla_\nu p &= -\frac{1}{2}\partial_\nu \ln\sqrt{-g_{00}},
\end{aligned}
\tag{4.1}
$$

where, the metric allows $g^{00} = \frac{1}{g_{00}}$.

As Landau points out [LL59], in the weak field limit where $(e+p) \approx \rho$ and $g_{00} = -1 - 2\phi$ with

$$\ln(1+2\phi) \approx 2\phi$$

Equation 4.1 reduces to

$$
\begin{aligned}
\frac{1}{\rho}\nabla P &= -\nabla\phi \\
\nabla P &= \rho \mathbf{g},
\end{aligned}
$$

which is the classical condition of hydrostatics.

4.1 Numerical energy transport

We can now write out the equations that must be obeyed by the relativistic fluid. For the purposes of this paper, we will reduce down the equations under certain assumptions.

1. One-dimensional fluid flow
2. Flat Minkowski space
3. The fluid is barotropic (i.e. P = C e, where C is a constant)

4. Energy density (e) is conserved within the medium.

   As the energy density is constant we can write out the following system of equations

$$\begin{aligned} \nabla_\mu u^\mu e &= 0 \\ (e+p)u^\mu \nabla_\mu u_\nu + u_\nu u^\mu \nabla_\mu p &= -\nabla_\nu p \\ p &= Ce, \end{aligned}$$

which, under the barotropic fluid assumption, reduce further to

$$\nabla_\mu (u^\mu e) = 0 \tag{4.2}$$

$$u^\mu \nabla_\mu u_\nu = \frac{-C}{(1+C)e} \nabla_\nu e. \tag{4.3}$$

From the Lorentz transforms, it can be shown that

$$t = \gamma\tau \qquad \text{where} \qquad \gamma = \left(1 - v^2\right)^{\frac{-1}{2}}.$$

Under this transformation, we find that

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma v^\mu.$$

Along with assumptions the remaining, Equations 4.2 and 4.3, become

$$\begin{aligned} \partial_t(\gamma e) + \partial_x(\gamma e v) &= 0 \\ \gamma \partial_t(\gamma v) + \partial_x\left(\frac{(\gamma v)^2}{2}\right) &= -\frac{C}{1+C}\partial_x \ln e. \end{aligned}$$

Using a spectral fourth order Runge-Kutta differencing scheme, we can attempt to solve these equations. The details of the method can be found in Duran's numerical methods text [Dur99]. The details have been omitted here to avoid confusion. Appendix A contains the code used to approximate the system of equations along with a brief description of the method.

In order to demonstrate the solution to this equation, we will use periodic boundary conditions. We will assume that all quantities are unit-less and we will implement initial conditions to represent a relativistic fluid with the energy density grouped into a dense region. A hyperbolic secant function was selected for the energy density function as it represents a single energy density packet centred around the origin. A relativistic velocity of $0.5c$ was selected, and a background energy of one was used to ensure a non-zero energy level throughout the domain.

$$\begin{aligned} u(x,0) &= 0.5 \\ e(x,0) &= \operatorname{sech}\left(\frac{x}{0.5}\right) + 1 \end{aligned}$$

A time step of $\Delta t = 1e-5$ and 512 grid points were used. Figure 4.1 outputs the results of the computation for three different values of $C = \{0, \frac{1}{3}, \frac{2}{3}\}$ corresponding to non-interacting matter, relativistic matter, and cold matter respectively. For details on these values, Battaner [Bat96] has a description of the statistical mechanical derivation.

The purpose to providing this numerical solution if three-fold. First, it demonstrates the complexity of the corresponding solution. It can be clearly seen that the interaction between the velocity and the energy density is very complicated. For C=0, there is no effect of the energy density on the velocity, and

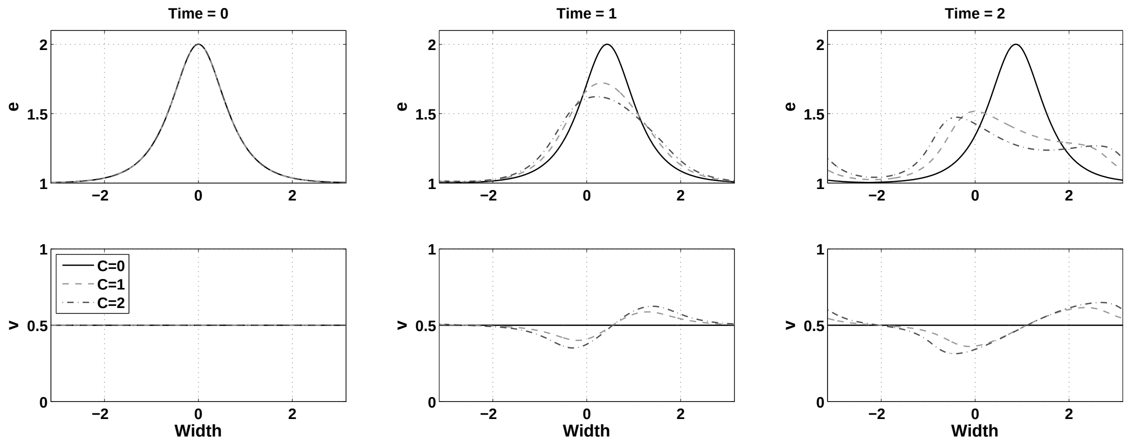

Figure 4.1: Energy transport relativistic hydrodynamics for values of P = $\{0e$ (solid line), $\frac{1}{3}e$ (dashed line), $\frac{2}{3}e$ (dash-dot line)$\}$ at time steps t=$\{0, 1, 2\}$ where both the energy density (e) and 3-velocity (v) have been output.

vice-verse. For $C > 0$, the change of one feeds back onto the other causing the solutions to distinguish themselves. Second, this serves as a basis upon which future work can be preformed. Thirdly, these numerics demonstrate the fact that, even under the simplifying assumptions of Minkowski space, the solution to the problem highly dependent upon the relativistic components of the equation. In this case, we see that the higher proportion of P to the energy density (i.e. larger values of C), the more rapid the transition from one state to another.

5 Spacetime Curvature

Up to this point we have assumed that the metric was known, that is the fluid does not substantially change the curvature of spacetime. This has many applications, however, it does leave something desired in order to get a more general theory. We return to the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}.$$

It is often convenient to convert this into the form

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}_{\alpha} \right). \tag{5.1}$$

These ten differential equations prove very difficult to solve. We can however, show that under certain symmetries, the system reduces to a simplified form.

For the purposes of this paper, we consider an application to a star. In reference to this, we will assume that the metric should be spherically symmetric. For the present purpose, let us also assume that the metric is constant in time. In reference to this, it can be shown that the most generic metric that can be written in spherical coordinates is

$$g_{\mu\nu} = \text{diag}\left[-B(r), A(r), r^2, r^2 \sin^2\theta\right]_{\mu\nu}, \tag{5.2}$$

for which we can easily calculate the connection coefficients.
The components of the Ricci tensor can be found using Equation 2.6. They are

$$\begin{aligned}
R_{00} &= -\frac{1}{2A}\frac{d^2B}{dr^2} + \frac{1}{4A}\frac{dB}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{r}\frac{1}{A}\frac{dB}{dr} \\
R_{rr} &= \frac{1}{2B}\frac{d^2B}{dr^2} - \frac{1}{4B}\frac{dB}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{r}\frac{1}{A}\frac{dA}{dr} \\
R_{\theta\theta} &= -1 + \frac{r}{2A}\left(-\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) + \frac{1}{A} \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\
\text{else} &= 0.
\end{aligned} \tag{5.3}$$

For a more detailed explanation, see Battaner [Bat96].

5.1 Schwarzschild metric

The previous assumptions prove reasonable when considering a star in space. In the region external to the star (provided the star is not rotating or charged) the stress energy tensor becomes null, and thus we must solve the complete set of equations

$$R_{\mu\mu} = 0.$$

The original solution to this problem was originally proposed by Schwarzschild in 1916 (the original article has recently been republished [Sch99]). Schwarzschild showed that the metric

$$g_{\mu\nu} = \text{diag}\left[-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta\right]_{\mu\nu}$$

is a solution.

5.2 Curvature Deformation

Inside the star, however, is a very different story. We will assume here that we still have spherical symmetry and the star is not rotating or charged. Many of the following details can be found in Battaner [Bat96]. We find that the basic form of the metric is the same as in (5.2), as such the Ricci tensor will in turn, have the same basic structure as Equation 5.3. However, now the field equations become

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha\right).$$

Recall that for a perfect fluid (Equation 3.1), we can write the stress energy tensor as

$$T^{\mu\nu} = (e + p)\, u^\mu u^\nu + p g^{\mu\nu}.$$

In the rest frame of the fluid we can write that

$$u_j = (-\sqrt{B}, 0, 0, 0).$$

Thus, the solution stress energy tensor becomes

$$T_{\mu\nu} = \text{diag}\left[eB, pA, pr^2, pr^2\sin^2\theta\right]. \tag{5.4}$$

We can then use the metric on Equation 5.4 to find

$$T^{\alpha}_{\alpha} = 3p - e. \tag{5.5}$$

Thus, Equations 5.4 and 5.5 combined with Equation 5.1 give, (Recall that we use units such that G=1),

$$\frac{R_{\mu\nu}}{8\pi} = \text{diag}\left[\frac{1}{2}(3p+e)B, -\frac{A}{2}(p-e), -\frac{r^2}{2}(p-e), -\frac{r^2\sin^2\theta}{2}(p-e)\right], \tag{5.6}$$

which, equated with Equation 5.3, provides a complete system of equations to solve for the components of the metric.

$$-\frac{1}{2A}\frac{d^2B}{dr^2} + \frac{1}{4A}\frac{dB}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{r}\frac{1}{A}\frac{dB}{dr} = -4\pi(3p+e)B \tag{5.7}$$

$$\frac{1}{2B}\frac{d^2B}{dr^2} - \frac{1}{4B}\frac{dB}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{r}\frac{1}{A}\frac{dA}{dr} = 4\pi A(p-e) \tag{5.8}$$

$$-1 + \frac{r}{2A}\left(-\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) + \frac{1}{A} = 4\pi r^2(p-e) \tag{5.9}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}. \tag{5.10}$$

Combining Equations 5.7 and 5.8 gives

$$-\frac{1}{r}\left(\frac{dB}{dr} + \frac{B}{A}\frac{dA}{dr}\right) = -8\pi AB(e+p), \tag{5.11}$$

using Equation 5.9,

$$\frac{d}{dr}\left(\frac{r}{A}\right) = 1 - 8\pi r^2 e. \tag{5.12}$$

Recall that here e is the energy density of the fluid, containing mass and internal energy, so we can write out that

$$U = \int_0^r 4\pi r^2 e dr$$
$$\frac{r}{A} = r - 2U$$
$$A = \left(1 - \frac{2U}{r}\right)^{-1}.$$

Keep in mind that we have two boundary conditions on A. At $r = 0$, we want to make sure the A is finite, and for $r > R$, the radius of the star, we require A to become the Schwarzschild value, $A_R = \left(1 - \frac{2M}{R}\right)^{-1}$.

Similarly, Equations 5.11 and 5.9 provide an equation for B

$$\frac{1}{B}\frac{dB}{dr} = \frac{2A}{r^2}\left(U + 4\pi r^3 p\right). \tag{5.13}$$

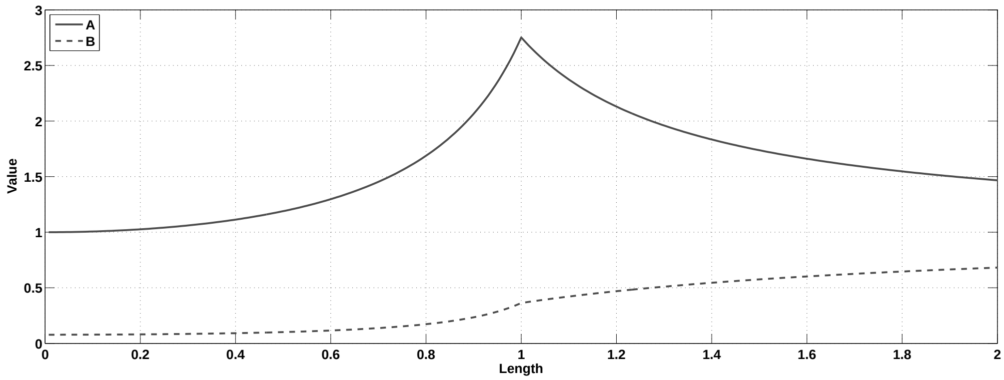

Figure 5.1: Solution to the A and B metric elements, assuming units such that the radius of the star is $R = 1$, and the mass is given as $M = 1/\pi$.

Here we can find a boundary condition such that, on the surface of the star, $p = 0$, $U = M$, thus, (where BA =1)

$$\frac{dB}{dr} = \frac{2M}{R^2}, \tag{5.14}$$

where again R is the radius of the star.

This calculation is meant to demonstrate how the stress-energy tensor modifies the curvature of space-time. In the domain without the nice symmetric properties we've introduced here, it is often necessary to solve the equations numerically. We refer the reader to further texts on the subject, such as the book written by Wilson [WM03]. These computations themselves prove very difficult and are omitted here.

5.3 Graphical solution

In order to understand the metric internal to the star, we consider the graph of A and B as a function of distance from the origin. For simplicity we assume that the density of the star is constant, and assume units such that the radius is 1 and mass is $1/\pi$. This is meant to provide a qualitative solution to the metric inside of a star. Figure 5.1 plots the resulting values of A and B under these conditions.

Notice that the solution is piecewise continuous at surface of the star ($r = 1$). Notice also that the A parameter becomes close to 1 near the centre, its asymptotic limit.

6 Conclusion

The current paper is meant to provide a brief introduction to relativistic fluids. As much as possible, this work has tried to compare the relativistic results with their Newtonian counterparts in order to provide basis for the new material. In here, the equations of motion of a perfect fluid have been written out and the static solution has been provided.

One major extension of relativistic hydrodynamics which we has not tackled here is Magnetohydrodynamics (MHD). MHD is the study of electrically charged fluids and has been applied to a wide variety of topics including stellar modelling. Golub [GP10] has a good classical approach to the topic. This is still a very active area of research.

For a more in depth discussion of relativistic hydrodynamics, the reader is referred to two well written texts on the subject. Andersson's discussion [AC07] provides a much more rigorous approach to the subject, and spends a great deal of time developing the mathematics of differential geometry. Also, Anile's book [Ani89] extends much of the above work to the case of MHD. There are many other well written texts on the subject, we simply provide the reader with two examples to serve as a basis for further research.

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A Numerical Hydrodynamics Methodology

This section discusses the code used to compute the solution to the relativistic fluid equations found in Section 4.1. The source code is available along side the online version of this article.

Putting the equation into the following form

$$\partial_t u = F(u),$$

this spectral method decomposes the function u_j, sampled at the grid points x_j, into its truncated Fourier series

$$u_j = \sum_{k=-\frac{(N+1)}{2}}^{\frac{N-1}{2}} a_k \exp ikx_j,$$

with N the number of grid points. Note that we have removed the $k = N/2$ wavenumber.

This technique allows us to use Matlab's built in FFT methods to compute derivatives of the corresponding function. Once the method for computing F has been established, we can then implement a

Fourth-Order Runge-Kutta method using

$$
\begin{aligned}
q_1 &= \Delta t F(u^n) \\
q_2 &= \Delta t F(u^n + \frac{q_1}{2}) \\
q_3 &= \Delta t F(u^n + \frac{q_2}{2}) \\
q_4 &= \Delta t F(u^n + q_3) \\
u^{n+1} &= u^n + \frac{q_1 + 2q_2 + 2q_3 + q_4}{6},
\end{aligned}
$$

where the superscripts, n, refer to the time step. The details of such a computation are included in Durran [Dur99].