A Combinatorial Approach to Finding Dirichlet Generating Function Identities

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ABSTRACT: This paper explores an integer partitions-based method for obtaining Dirichlet generating function identities. In the process we shall generalize a previous result, obtain previously unknown formulae for the Möbius and Liouville Dirichlet generating functions, and obtain a formula on unit fractions.

1 Introduction

The positive integers can be expressed as sums of positive integers in many ways. For example: $6 = 3 + 3 = 1 + 5 = 2 + 2 + 2$, where 6 is also a partition of 6. The problems concerning such a representation are additive in nature. If order matters, the sum is called a composition. If the order of the summands does not matter, the sum is called a partition. Pak [Pak09] gives a history of partition identities and an overview of various constructions that are used to obtain those identities.

One could also think about the multiplicative properties of the integers. Each positive integer can be represented as a product of positive integers. Each such representation is a multiplicative partition. For example, $24 = 2 \times 12 = 4 \times 6$, where 24 is also its own multiplicative partition. The set of numbers in such a representation is called a factorization. If the order of the summands matters, the factorization is called ordered. When order is immaterial, the factorization is called unordered. Therefore unordered factorizations are the multiplicative equivalent of integer partitions. Sometimes we shall call them multiplicative partitions as well.

Knopfmacher and Mays [KM05, KM03] give results about factorizations. They use algebraic manipulation and bijections to obtain factorization identities. The goal of this paper is to show that symbolic methods, like the ones used by Pak, can be applied with some modifications to obtaining multiplicative partition identities. We will essentially follow both papers and generalize some of the ideas presented. The following two identities appear in the process

$$\frac{\zeta(2s)}{\zeta(s)} = 1 - \sum_{k=1}^{\infty} \frac{p_k^{-s}}{(1+2^{-s})(1+3^{-s})(1+5^{-s})\dots(1+p_k^{-s})}$$

$$\frac{1}{\zeta(s)} = 1 - \sum_{k=1}^{\infty} p_k^{-s}(1-2^{-s})(1-3^{-s})(1-5^{-s})\dots(1-p_{k-1}^{-s}),$$

where p_k is the k-th prime. These are Dirichlet generating functions for the Möbius and Liouville functions from number theory. The identities are a beautiful complement to the following two identities, obtained by Knopfmacher and Mays

$$\zeta(s) = 1 + \sum_{k=1}^{\infty} \frac{p_k^{-s}}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\dots(1-p_k^{-s})}$$

$$\frac{\zeta(s)}{\zeta(2s)} = 1 + \sum_{k=1}^{\infty}(1+2^{-s})(1+3^{-s})(1+5^{-s})\dots(1+p_{k-1}^{-s}).$$

In Section 2 we will introduce the notation that will be used throughout the rest of this paper.

We assume familiarity with Dirichlet generating functions (DGF's). The section will conclude with two basic examples that are important in the study of factorizations. In 2003, Knopfmacher and Mays [KM05] derived several identities for factorization generating functions. For example, the Dirichlet generating function for unordered factorizations and unordered factorizations with distinct divisors are

$$F(s) = \prod_{n=2}^{\infty} \frac{1}{1-n^{-s}}, \quad F_d(s) = \prod_{n=2}^{\infty} (1+n^{-s}).$$

Also, they derived the following DGF for the number of unordered factorizations with largest divisor k

$$F(s) = \frac{k^{-s}}{(1-2^{-s})(1-3^{-s})\dots(1-k^{-s})}.$$

In Section 3 we will use methods of Knopfmacher and Mays [KM05] to treat the general case of an arbitrary environment. In the end of the section we shall derive some new identities not mentioned in said paper. One of those identities will be the DGF for the number of unordered factorizations with smallest divisor k. Based on this identity, we will derive some new identities for two number theoretic DGF's (the Möbius and Liouville functions).

In Section 4 we shall use the symbolic method to obtain a new factorization identity. It is based on the idea of the Durfee square that is used for partitions. In the process we will also obtain an identity on unit fractions.

We would like to thank the reviewers of *The Waterloo Mathematics Review* for catching many errors and making many useful suggestions for improving the exposition of this paper.

2 Unordered Factorizations

We will model our notation after the notation used by Pak [Pak09].

Definition 2.1. Let an *unordered factorization* μ be an integer sequence

$$(\mu_1, \mu_2, \dots, \mu_L), \quad \text{where} \quad \mu_1 \geq \mu_2 \geq \dots \geq \mu_L.$$

The μ_i are the *parts* of the factorization. The *size* of the factorization is

$$|\mu| = \prod_n \mu_n.$$

The *length* is the number of distinct entries in μ (denoted by $L(\mu)$). If $|\mu| = n$, μ is an *unordered factorization* of n (denoted by $\mu \vdash n$). Let $a(\mu)$ and $s(\mu)$ be the largest and smallest parts of μ. The *multiplicity* m_d of an integer d is the number of times it is present in μ. We can also use the notation $\mu = (1^{m_1}, 2^{m_2}, \dots)$.

For example, let $n = 1567641600 = 2^{12}3^7 5^2 7^1$. One unrestricted unordered factorization is

$$\mu = (2^4, 3^3, 4^4, 5^2, 7, 9^2).$$

We can also use the fundamental theorem of arithmetic and decompose n in primes. One such factorization is

$$\nu = (2^{12}, 3^7, 5^2, 7^1),$$

Next

$$a(\mu) = 9, \quad s(\mu) = 2$$
$$a(\nu) = 7, \quad s(\nu) = 2.$$

When considering the additive problem for integer partitions, one is quickly confronted by Ferrers diagrams. For example, they are introduced in Pak's paper under the name Young diagrams. They are not to be confused with Young tableaux. A Ferrers diagram is a collection of squares on $\mathbb{Z}^2$ that represent an integer partition. There are different conventions but the important part is that one axis encodes size and the other encodes multiplicities. We model the definition after the one for Young diagrams used by Pak [Pak09]. Multiplicatively, such a diagram does not work and we need to modify it.

Definition 2.2. A *multiplicative partition diagram* $[\mu]$ of an unordered factorization $\mu \vdash n$ is a collection of 1×1 squares (i, j) on a square Cartesian grid. Let $1 \leq i \leq L(\mu)$ and $1 \leq j \leq m_i$. The parts of the partition are on the i-axis and their multiplicities are on the j-axis. The *complete* diagram $[\mu]_o$ is the multiplicative partition diagram where we have an empty space for every 0 power of a prime.

Note that these diagrams are similar to composition diagrams but have an important difference. In a composition diagram, the horizontal axis encodes the size of the element itself. That is, 5 squares indicate the integer 5. In a multiplicative partition diagram, the horizontal axis encodes multiplicities. That is, 5 squares for a divisor d indicate that d occurs 5 times in the particular factorization of the integer whose diagram we are looking at. Also, the first row of a diagram of an integer n encodes the multiplicity of the smallest positive integer d dividing n, $d \geq 2$. For μ and ν, the diagrams are

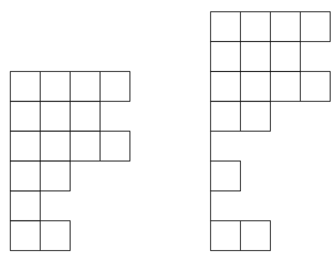

Figure 2.1: $[\mu]$ and $[\mu]_o$ for $\mu = (2^4, 3^3, 4^4, 5^2, 7, 9^2)$

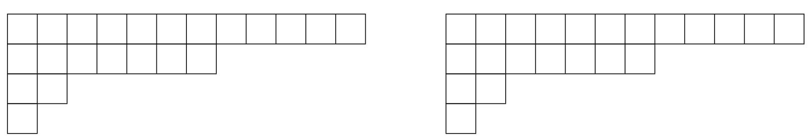

Figure 2.2: $[\nu]$ and $[\nu]_o$ for $\nu = (2^{12}, 3^7, 5^2, 7^1)$. The latter does not have empty spaces because we restrict to prime divisors.

Next, we would like to focus on certain subsets of the positive integers, i.e. the prime numbers, the residues of a certain modulo, etc. Also we would like to focus on certain allowed multiplicities of parts in the multiplicative partition, i.e. distinct divisors, multiplicities from a certain residue class, other finite sets, etc. Here we define a unifying structure for all the different sets of divisors and allowed multiplicities.

Definition 2.3. Let $\mathcal{D}$ be the *set of allowed divisors.* For $d \in \mathcal{D}$, let $\mathcal{M}_d$ be the *set of allowed multiplicities for d.* The *environment* we are working in is the pair $(\mathcal{D}, \mathcal{M})$.

For $\mathcal{M}_d$ and $\mathcal{D}$ we can have the positive integers ≥ 2, the prime numbers, denoted by $\mathbb{P}$ and the integers equivalent to b modulo a. Also, $\mathcal{M}$ is just the collection of sets of allowed multiplicities for the divisors in $\mathcal{D}$. We shall use generating functions to extract information on multiplicative problems. When considering additive partitions, the most often used generating function is the ordinary generating function (OGF for short). For multiplicative problems the DGF is the natural tool.

The most famous DGF is the Riemann Zeta function (the DGF for the sequence $(1, 1, \ldots)$), given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}}.$$

The second equality is the Euler product representation for the zeta function. Both sides of the equation converge for s with real part greater than 1. In this paper, we will treat the DGF's formally, so their convergence properties will not be covered. This could be the topic of future work.

Let's start with a motivating example that is covered by Knopfmacher and Mays [KM05]. Let the allowed divisors be the positive integers except 1. For each divisor, let the set of multiplicities be the set of non-negative integers. That is $\mathcal{D} = \mathbb{N}\backslash\{1\}$ and $\mathcal{M}_n = \mathcal{M} = \mathbb{N} \cup \{0\}$ for each divisor n. Next, let $f(n)$ be the number of multiplicative partitions of n with divisors from $\mathcal{D}$. Note that $f(1)$ is thus not defined. For convenience, we define it to be $f(1) = 1$. Our first goal is to find the DGF for the sequence $(f(n))$. Formula (3) [KM05] gives us the following DGF

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{n=2}^{\infty} \frac{1}{1 - n^{-s}}.$$

Using the geometric series, we can expand the right hand side (RHS) as

$$\prod_{t=2}^{\infty} \frac{1}{1 - t^{-s}} = \prod_{t=2}^{\infty} \sum_{k=0}^{\infty} t^{-ks} = \prod_{t=2}^{\infty} \sum_{k=0}^{\infty} \left(t^k\right)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where the coefficients c_n count the number of ways we have arrived at n^{-s} through various multiplications. This is exactly the number of unordered factorizations of n. Note that the product runs through all the elements in the set $\mathcal{D}$. Also, the n-th term in the product can be expanded as a geometric sum. Each of those sums runs over the set of allowed multiplicities $\mathcal{M}_n$. Next we consider the following product

$$F_d(s) = \prod_{k=2}^{\infty} (1 + k^{-s}).$$

We can expand the product as in the previous example. If c_n is the coefficient of n^{-s} in the expansion of $F_d(s)$, it again counts unordered factorizations. However, the situation is different this time. For each k of the product, we can choose either k^{-s} or 1 in the expansion. This corresponds to k either being an element of $\mu \vdash n$ or not. However it cannot be an element more than once. Hence, the multiplicity of k is restricted to 0 and 1. This is reflected in the k-th term in the product. From the preceding discussion, $F_d(s)$ is the DGF for the number of unordered factorizations with distinct parts. Also, the product forces us to define $f(1) = 1$ for this problem as well. We could define $f(1) = 1$ but this will put restrictions on what we can do. It is more useful to allow it to be defined by the problem.

Naturally, if we want to restrict ourselves to a finite set of allowed divisors, all we need to do is truncate the product and consider terms only from $\mathcal{D}$. Similarly, we can do the same on each term of the product.

3 General approach

We will now generalize the above discussion. Also, we will use the multiplicative factorization diagrams to our advantage. We begin by writing down another definition

Definition 3.1. Let $(\mathcal{D}, \mathcal{M})$ be an environment. Let C be a constraint. Then n is *reachable* if there is at least one $\mu \vdash n$ from the environment under the constraints C. Let $f(n, C)$ count the number of unordered factorizations of n. Let $f(n) = f(n, \emptyset)$ (unconstrained). Let $\mathcal{M}_d(s)$ be the DGF for the constraints for the divisor d. Then we define

$$F(s;C) = \sum_{n=1}^{\infty} \frac{f(n,C)}{n^s}, \quad M_d(s) = \sum_{m\in\mathcal{M}_d} \frac{1}{d^{ms}}.$$

We also write $F(s;\emptyset) = F(s)$.

The kinds of constraints we will focus on are simple—divisors and/or multiplicities will be restricted in size or number. For example, we can set the largest admissible divisor to some integer or we can set the maximum number of occurrences of a divisor. In the above definition, one could ask why are we taking the sum over all n when all we need is $d \in \mathcal{D}$. The reason is that product identities for $F(s;C)$ may give terms n^{-s}, unreachable in our environment. Therefore, the coefficients $f(n,C)$ for such terms must be defined through the product. For example, we set $f(1) = 1$ in the determination of the DGF for all multiplicative partitions. In that example, the only element unreachable from our environment is $n = 1$. This is true in general—either $f(1) = 0$ or we have to define it as $f(1) = 1$. There is no other case of unreachable elements n, for which we have to define $f(n)$. This is not covered by Knopfmacher and Mays in 2003 [KM05]. Next, we have our result about a general environment. We were unable to find a previous proof of this result in the literature

Proposition 3.1. Let $(\mathcal{D}, \mathcal{M})$ be an environment. Then the generating function for $f(n)$ is

$$F(s) = \prod_{d\in\mathcal{D}} M_d(s)$$

The only unreachable element that we may need to define $f(n)$ for is $n = 1$. We have to set $f(1) = 1$ if and only if 0 is in $\mathcal{M}_d(s)$ for every $d \in \mathcal{D}$.

Proof. As in the discussion above, we need to formally expand the product on the RHS. We use the definition for $\mathcal{M}_d(s)$

$$\prod_{d\in\mathcal{D}} M_d(s) = \prod_{d\in\mathcal{D}} \sum_{m\in\mathcal{M}_d} d^{-ms} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

The coefficient c_n counts the number of times that elements from the product multiply to n^{-s}. If $c_n = 0$, then n is not reachable, since it is not the product of elements from the environment. If $c_n \geq 1$ we have two cases. If $n \geq 2$, there must be some $d \in \mathcal{D}$, $d \geq 2$ with allowed multiplicity greater than 0, such that $d|n$. Therefore, n is reachable and $c_n = f(n)$. Since all divisors are ≥ 1, if $c_n = 1$, this means that $0 \in \mathcal{M}_d$ for every d. Otherwise, we would never reach it. So in this case we need to set $f(1) = 1$ for our environment.

Now, suppose that $f(1) = 1$ and the latter half of the statement is not true. Then there is a d such that $0 \notin \mathcal{M}_d$. Then there is no way we could get $c_1 > 0$, since every product in the expansion of the RHS would have a factor of at least d^{-s}. Then this is a contradiction and the proof is complete. □

The proposition allows us to construct the DGF for an environment $(\mathcal{D}, \mathcal{M})$ in an intuitive and efficient way. For example, if we are unrestricted or if we use only even or distinct divisors we get the DGF's

$$F(s) = \prod_{n=1}^{\infty} \frac{1}{1-n^{-s}}, \quad F_e(s) = \prod_{n=1}^{\infty} \frac{1}{1-(2n)^{-s}}, \quad F_d(s) = \prod_{n=2}^{\infty} (1+n^{-s}).$$

If we only want to use a finite subset of divisors, we just truncate the products. Also, we can readily construct hybrids. The only thing left to do is to check whether 0 is in the allowed multiplicities for each $d \in \mathcal{D}$. This is the case for all the identities so far. This scheme is very similar to the scheme for partitions. The main difference is that in partitions we use OGF's and in the multiplicative case we use DGF's. For example, if we are unrestricted or if we use only even or distinct summands, we get the OGF's

$$P(s) = \prod_{n=1}^{\infty} \frac{1}{1-z^n}, \quad P_e(s) = \prod_{n=1}^{\infty} \frac{1}{1-z^{2n}}, \quad P_d(s) = \prod_{n=1}^{\infty} (1+z^n).$$

We shall look at some simple constraints. The multiplicative partition diagram can help us visualize some transformations and help us derive identities. The first constraint we can look at is that of greatest or smallest elements. The greatest element constraint is covered for a few cases by Knopfmacher and Mays [KM05]. However, the minimum element constraint is not covered. We will do so here and we will obtain some interesting identities. If we have the constraint $C : a(\mu) \leq d$, then we are essentially just truncating $\mathcal{D}$ to elements smaller than or equal to d. If $C : s(a) \geq d$, we have a truncation from below. Similarly one can consider a constraint on the multiplicity of an element. The truncation follows there as well.

Proposition 3.2. Let $(\mathcal{D}, \mathcal{M})$ be an environment. Then

$$F(s; a(\mu) \leq u) = \prod_{d \in \mathcal{D}, d \leq u} M_d(s), \quad F(s; s(\mu) \geq u) = \prod_{d \in \mathcal{D}, d \geq u} M_d(s).$$

Proof. The proof follows from the previous discussion and Proposition 3.1. □

There is an analogous result for partitions. In Section 5 [KM05] Knopfmacher and Mays perform a largest element decomposition. That is, they find the generating function for unordered factorizations of n with largest factor k. We shall imitate their exposition from the point of view of environments and then we shall generalize it. Let $\mathcal{D} = \mathbb{N} \backslash \{1\}$ and $\mathcal{M}_d = \mathbb{N} \cup \{0\}$ for all d in $\mathcal{D}$. We use the constraint $C : a(\mu) = k$. Let n be a positive integer divisible by k. From the paper, the factorization of n "can be written as $k \times a$ where a represents a factorization of n/k into factors $\leq k$". Then authors give the generating function

$$F(s; a(\mu) = k) = \sum_{n=1}^{\infty} \frac{f(n, a(\mu) = k)}{n^s} = \frac{k^{-s}}{(1 - 2^{-s})(1 - 3^{-s}) \ldots (1 - k^{-s})},$$

where we have written it in the new notation. Looking at this function, we can see that the right-hand side is just $k^{-s} F(s; a(\mu) \leq k)$. Next the authors sum the above identity over all k and obtain the identity

$$\prod_{n=2}^{\infty} \frac{1}{1 - n^{-s}} = 1 + \sum_{k=2}^{\infty} \frac{k^{-s}}{(1 - 2^{-s})(1 - 3^{-s}) \ldots (1 - k^{-s})}.$$

In the later part of that section they reapply this method to derive more "Series—Product" identities of similar form. We shall generalize this method and extend it. To do so we will go through the same argument in an arbitrary environment. However, we have to modify the approach a little bit because when we divide by k, we cannot be sure if the remaining multiplicity of k is allowed. The following diagrams show what an arbitrary multiplicative diagram looks like with maximum and minimum element constraints. Note that the bars for u can be of any length. We start with the discussion for the maximum element decomposition and later we will just state the results for the minimum element decomposition without proof because the proofs are analogous.

Proposition 3.3. Let $(\mathcal{D}, \mathcal{M})$ be an environment. Let the largest part of μ be u and let it have multiplicity v (denote this by $C_u^v : a(\mu) = u, m_u = v$). Then

$$F(s, C_u^v) = u^{-vs} \prod_{d \in \mathcal{D}, d < u} M_d(s).$$

Furthermore,

$$F(s, a(\mu) = u) = \prod_{d < u, d \in \mathcal{D}} M_d(s) \sum_{m \in \mathcal{M}_u, m \neq 0} u^{-ms}.$$

If $0 \notin \mathcal{M}_u$, then

$$F(s, a(\mu) = u) = \prod_{d \leq u, d \in \mathcal{D}} M_d(s).$$

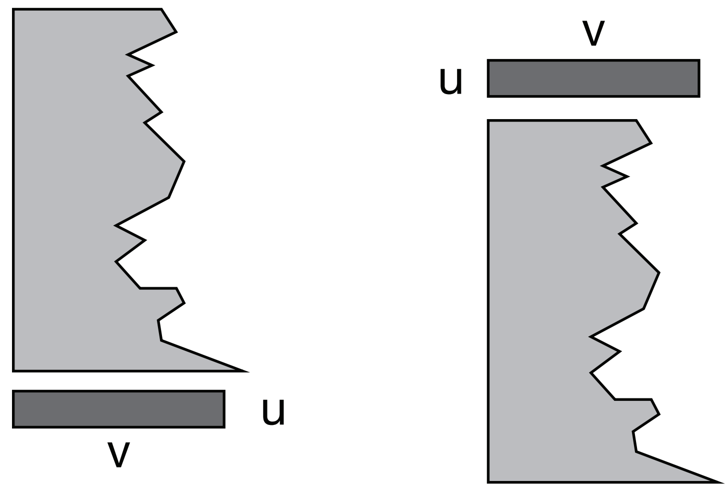

Figure 3.1: Constraints $a(\mu) = u$ and $s(\mu) = u$ with multiplicity v

If $0 \in \mathcal{M}_u$, then

$$F(s, a(\mu) = u) = (M_u(s) - 1) \prod_{d<u, d\in\mathcal{D}} M_d(s).$$

Proof. Suppose that $\mu \vdash n$ is a multiplicative partition satisfying condition C_u^v. Its largest part is u and it has multiplicity v. All multiplicative partitions of n contain v copies of u. We can map each partition of n to a partition of n/u^v by removing u^v. Furthermore this map is a bijection between the multiplicative partition diagrams of n and those of n/u^v. Thus $f(n, C_u^v) = f(n/u^v, a(\mu) < u)$ and we can perform the following summation

$$\begin{aligned} F(s, C_u^v) = \sum_{n=1}^{\infty} \frac{f(n, C_u^v)}{n^s} &= \sum_{n=1}^{\infty} \frac{f(n/u^v, a(\mu) < u)}{(\frac{n}{u^v})^s (u^v)^s} \\ &= u^{-vs} \sum_{n=1}^{\infty} \frac{f(n, a(\mu) < u)}{n^s} = u^{-vs} F(s, a(\mu) < u) \\ &= u^{-vs} \prod_{d\in\mathcal{D}, d<u} M_d(s) \end{aligned}$$

Next, we need to prove the second identity. If μ is a multiplicative partition and $u \in \mu$, then u has some multiplicity. The classes of multiplicative partitions containing μ are disjoint with respect to multiplicity. Therefore, the DGF for all μ containing u is just the sum $F(s, C_u^m)$ over all non-zero multiplicities m in $\mathcal{M}_u$ the identity follows directly. If $0 \notin \mathcal{M}_u$, then

$$\sum_{m\in\mathcal{M}_u, m\neq 0} u^{-ms} = \sum_{m\in\mathcal{M}_u} u^{-ms} = M_u(s).$$

If $0 \in \mathcal{M}_u$, then

$$\sum_{m\in\mathcal{M}_u, m\neq 0} u^{-ms} = -1 + \sum_{m\in\mathcal{M}_u} u^{-ms} = M_u(s) - 1.$$

□

For the example before the proposition, we have that for d, $\mathcal{M}_d$ is the set all possible multiplicities. Therefore

$$F(s,a(\mu)=k)=\left(\frac{1}{1-k^{-s}}-1\right)\prod_{t=2}^{k-1}\frac{1}{1-t^{-s}}=k^{-s}\prod_{t=2}^{k}\frac{1}{1-t^{-s}},$$

which is what we had before. Similarly, if our divisors are distinct, we have

$$F(s,a(\mu)=k)=(1+k^{-s}-1)\prod_{t=2}^{k-1}(1+t^{-s})=k^{-s}\prod_{t=2}^{k-1}(1+t^{-s}),$$

giving us the product inside the summation sign in Identity 28 of Knopfmacher and Mays [KM05]. Similarly, we can perform the same analysis on unordered factorizations on primes to give us Identity 29 of said paper [KM05]

$$F(s,a(\mu)=p_k)=\frac{p_k^{-s}}{(1-2^{-s})(1-3^{-s})\dots(1-p_k^{-s})}.$$

Thus, Proposition 8 unifies and generalizes the approach of Knopfmacher and Mays. The next step is to get identities like Identities 27, 28, 30, and 31 from Knopfmacher and Mays [KM05] right away.

Theorem 3.4. Let $(\mathcal{D},\mathcal{M})$ be an environment. Then

$$F(s)=c+\sum_{d\in D}F(s,a(\mu)=d),$$

where $c=1$ if 1 or 0.

Proof. To prove this we need to look at both sides of the equation and compare. Let $n>1$. Each $F(s,a(\mu)=d)$ enumerates all multiplicative partitions $\mu\vdash n$ that contain a largest factor d. Therefore, for different d, they enumerate disjoint classes. Since we are summing over all $d\in\mathcal{D}$, we are summing over all possible largest factors of μ. Hence the sum on the RHS enumerates all multiplicative partitions of n. The function on the LHS is just the DGF for all multiplicative partitions of n in the current environment. Hence the two sides formally expand to the same sums of n^{-s} for $n>1$. For the case $n=1$, if the LHS contains a non-zero coefficient for n^{-s}, we just need to add it to the RHS. By proposition 3.1, this coefficient can only be 1. If the coefficient of 1^{-s} is zero, then $c=0$. ☐

Thus Identities 27, 28, 30, and 31 from Knopfmacher and Mays [KM05] fall out of this formula right away. Here are Identities 30 and 31

$$\zeta(s)=1+\sum_{k=1}^{\infty}\frac{p_k^{-s}}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\dots(1-p_k^{-s})}$$
$$\frac{\zeta(s)}{\zeta(2s)}=1+\sum_{k=1}^{\infty}(1+2^{-s})(1+3^{-s})(1+5^{-s})\dots(1+p_{k-1}^{-s}).$$

Similar analysis can be done considering the smallest element of the multiplicative partition. The following theorem sums up the new results.

Theorem 3.5. Let $(\mathcal{D},\mathcal{M})$ be an environment. Let $S_u^v: s(\mu)=u, m_u=v$, that is we have the constraint that the smallest part of μ is u and it has multiplicity v. Then

$$F(s,S_u^v)=u^{-vs}\prod_{d\in\mathcal{D},d>u}M_d(s).$$

Furthermore,

$$F(s,s(\mu)=u)=\prod_{d>u,d\in\mathcal{D}}M_d(s)\sum_{m\in\mathcal{M}_u,m\neq 0}u^{-ms}.$$

Also

$$F(s) = c + \sum_{d \in D} F(s, s(\mu) = d),$$

where $c = 1$ is 1 or 0.

This scheme leads to some interesting consequences. The identity for $F(s)$ from the proposition can be re-written. First,

$$\begin{aligned} F(s, s(\mu) = d) &= \prod_{d<u, d \in \mathcal{D}} M_d(s) \sum_{m \in \mathcal{M}_u, m \neq 0} u^{-ms} \\ &= F(s) \left(\prod_{d \geq u, d \in \mathcal{D}} M_d(s) \right)^{-1} \sum_{m \in \mathcal{M}_u, m \neq 0} u^{-ms} \\ &= \frac{F(s)}{F(s, s(\mu) \geq d)} \sum_{m \in \mathcal{M}_u, m \neq 0} u^{-ms}. \end{aligned}$$

If $c = 1$, we get

$$F(s, s(\mu) = d) = \frac{F(s)(M_d(s) - 1)}{F(s, s(\mu) \geq d)}.$$

Then the identity from the last proposition can be re-written as

$$F(s) = 1 + \sum_{d \in \mathcal{D}} F(s, s(\mu) = d) = 1 + F(s) \sum_{d \in \mathcal{D}} \frac{(M_d(s) - 1)}{F(s, s(\mu) \geq d)},$$

giving us the identity

$$\frac{1}{F(s)} = 1 - \sum_{d \in \mathcal{D}} \frac{(M_d(s) - 1)}{F(s, s(\mu) \geq d)}.$$

Next, if $c = 0$, we get

$$F(s, s(\mu) = d) = \frac{F(s)}{F(s(\mu) > d)}$$

and

$$\sum_{d \in \mathcal{D}} \frac{1}{F(s, s(\mu) > d)} = 1.$$

The propositions lead to some interesting identities

$$\begin{aligned} \frac{1}{\zeta(s)} &= 1 - \sum_{k=1}^{\infty} \frac{p_k^{-s}}{(1+2^{-s})(1+3^{-s})(1+5^{-s}) \ldots (1+p_k^{-s})} \\ \frac{\zeta(2s)}{\zeta(s)} &= 1 - \sum_{k=1}^{\infty} p_k^{-s}(1-2^{-s})(1-3^{-s})(1-5^{-s}) \ldots (1-p_{k-1}^{-s}). \end{aligned}$$

They are the DGF's for the Möbius and Liouville functions from number theory. Note the beautiful symmetry between those identities and Identities 30 and 31 from Knopfmacher and Mays [KM05].

Now that we have gone over the general approach we may continue drawing analogies between additive and multiplicative partitions. We may pick many of the constructions given by Pak and transform them into multiplicative analogs but the one for the Durfee square seems simple enough to consider without difficulties, yet complex enough to illustrate the method.

4 Durfee Square equivalent

A Durfee square is the largest square in a Ferrers diagram. We can define a similar object for multiplicative partitions. The *largest square (rectangle)* in a complete multiplicative partition diagram is the largest square (rectangle) rooted at the top-left corner that fits in the diagram. In the case of rectangles, largest refers to area.

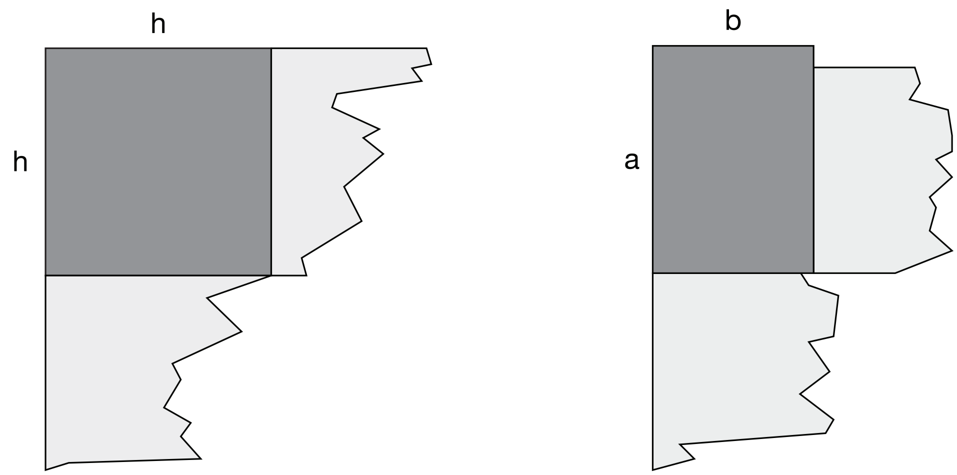

Figure 4.1: A largest square and a largest rectangle.

This can be done for a general environment but is messy. For the unrestricted environment we have two cases. Since the square is the largest, there must be some kind of obstruction on one of its sides. Let's consider two cases. Either $h+2$ has multiplicity $\leq h$ or $\geq h+1$. In the first case, the obstruction is on the lower side of the square. However, it could also be on the right. In the second case the obstruction is strictly on the right. The following diagram shows the two cases. We can look at parts A and B as environments

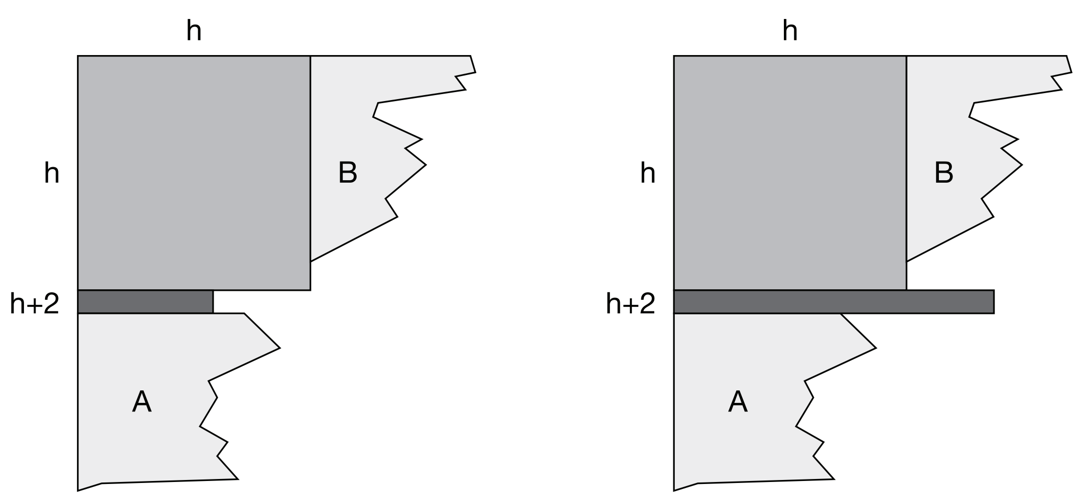

Figure 4.2: The two cases for a largest square.

with some restrictions. We decompose the problem into smaller problems for each case.

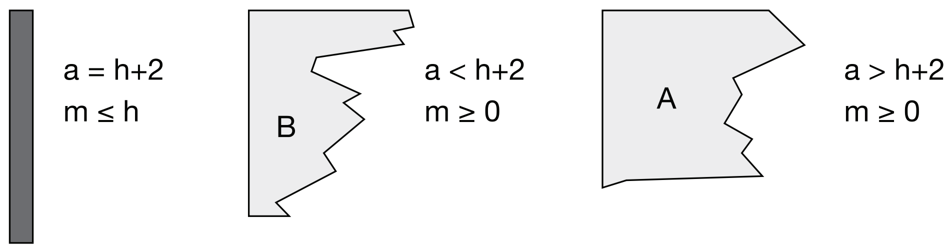

Figure 4.3: Case one.

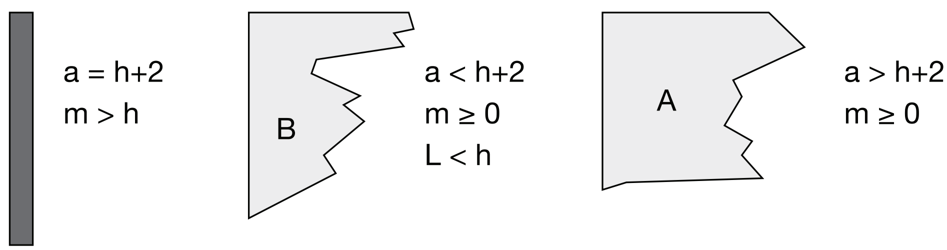

Figure 4.4: Case two.

Let's begin with the first case. The parts of μ in A and B have unrestricted multiplicity. Also A, B, the square, and part $h+2$ are independent of each other. The analysis so far gives us a bijection from the whole diagram to its smaller parts. That is we have a bijection taking the original partition μ to partitions of A, B, and part $h+2$. Since they are independent, we will multiply the generating functions together. First, the DGF for the $h+2$ part is

$$E_1(s) = \sum_{m=0}^{h} (h+2)^{-ms} = \frac{(h+2)^{-hs} - (h+2)^{s}}{1-(h+2)^{s}}.$$

The generating function for the square is

$$F_{hh}(s)^h = \prod_{n=2}^{h+1} n^{-hs} = ((h+1)!)^{-hs}.$$

The DGF's for A and B are

$$A(s) = \prod_{n=h+3}^{\infty} \frac{1}{1-n^{-s}}, \quad B_1(s) = \prod_{n=2}^{h+1} \frac{1}{1-n^{-s}}.$$

Note that

$$A(s)B_1(s) = (1-(h+2)^{-s}) \prod_{n=2}^{\infty} \frac{1}{1-n^{-s}} = (1-(h+2)^{-s})F(s).$$

Putting everything together, the generating function for this case is

$$F_1(s) = A(s)B_1(s)E_1(s)F_{hh}(s)^h.$$

Next, we work on the second case. Here A and the square have the same generating functions as in the first case. The $h+2$ part has the generating function

$$E_2(s) = \sum_{m=h+1}^{\infty} (h+2)^{-ms} = \frac{(h+2)^{-(h+1)s}}{1-(h+2)^{-s}}.$$

We need to be more careful with the DGF for B. We want to find the DGF for multiplicative partitions where the allowed divisors are from 2 to $h+1$. Also, at least one of those divisors must have multiplicity 0. If we consider the set of all allowed multiplicities, we can get the set where at least one of the divisors has multiplicity 0 by taking the whole set and subtracting the set where none of the divisors has multiplicity 0. In terms of the DGF's, this is

$$\begin{aligned} B_2(s) &= \prod_{n=2}^{h+1} \frac{1}{1-n^{-s}} - \prod_{n=2}^{h+1} \frac{n^{-s}}{1-n^{-s}} = B_1(s) - B_1(s) \prod_{n=2}^{h+1} n^{-s} \\ &= B_1(s)(1-F_{hh}(s)). \end{aligned}$$

Thus the DGF for the second case is

$$A(s)B_1(s)(1-F_{hh}(s))E_2(s)F_{hh}(s)^h.$$

Overall, since the cases are disjoint and all-including, the DGF for unordered factorizations with largest square $h \times h$ is

$$\begin{aligned} F_h(s) = F_1(s) + F_2(s) &= A(s)B_1(s)F_{hh}(s)^h(E_1(s) + (1-F_{hh}(s))E_2(s)) \\ &= (1-(h+2)^{-s})F(s)F_{hh}(s)^h(E_1(s) + (1-F_{hh}(s))E_2(s)). \end{aligned}$$

We can simplify the expression in the brackets a bit. It is

$$\begin{aligned} E_1(s) + E_2(s) - F_{hh}(s)E_2(s) &= \sum_{n=0}^{\infty} (h+2)^{-sm} - F_{hh}(s)E_2(s) \\ &= \frac{1}{1-(h+2)^{-s}} - \frac{(h+2)^{-(h+1)s}}{1-(h+2)^{-s}}((h+1)!)^{-s} \\ &= \frac{1}{1-(h+2)^{-s}} - \frac{(h+2)^{-hs}}{1-(h+2)^{-s}}((h+2)!)^{-s} \\ &= \frac{1-((h+2)!)^{-s}(h+2)^{-hs}}{1-(h+2)^{-s}}. \end{aligned}$$

Then $F_h(s)$ becomes

$$F_h(s) = \frac{F(s)}{((h+1)!)^{hs}} \left(1 - \frac{1}{((h+2)!)^s(h+2)^{hs}}\right).$$

The size h of the square can be any non-negative number. The size of the largest square defines equivalence classes among the multiplicative partitions. Hence if we sum $F_h(s)$ over all h, we will get $F(s)$ back. After cancelling $F(s)$ from both sides we have the following curious identity

Proposition 4.1. Formally, the following identity holds

$$\sum_{h=0}^{\infty} \frac{1}{((h+1)!)^{hs}} \left(1 - \frac{1}{((h+2)!)^s(h+2)^{hs}}\right) = 1.$$

Analytically, it seems that the identity holds for complex s with real part greater than 0. We have the following proposition about the partial sums of the above identity.

Proposition 4.2. If $S_N(s)$ is the partial summation of the above identity, then

$$S_N(s) = 1 - (N+2)!^{-(N+1)s}.$$

The formula is easy to prove by induction. The first few partial sums are

$$S_0(s) = 1 - 2^{-s},$$
$$S_1(s) = 1 - 36^{-s},$$
$$S_2(s) = 1 - 13824^{-s}.$$

5 Conclusion

As one can see from the survey by Pak, there are numerous constructions for additive partition identities. Yet, it seems like there are not that many constructions for multiplicative partition identities. The goal is to change this and in this paper we introduced diagrams and general theory that will be useful in this pursuit. Finally, as the Durfee square equivalent shows, this may not be straightforward but there are many other constructions that one can consider. Analyzing the multiplicative equivalents for other constructions could be the topic of future work.

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