Asymptotic eigenvalue distribution of random lifts

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October 29, 2012

Abstract: A *random n-lift* $L_n(G)$ of a base graph G is obtained by replacing each vertex v_i of G by a set V_i of n vertices, and generating a random matching between V_i and V_j for each edge $(v_i, v_j) \in G$. We show that the spectral density of a random lift $L_n(G)$ approaches that of a tree as we increase n by showing that the expected number of short cycles of length k in $L_n(G)$ (denoted $Z_k(G)$) tends to a constant λ_k. Moreover, we show that $Z_k(G)$ is Poisson distributed with parameter λ_k. We also give experimental results of the level spacing distributions and compare them to the Gaussian Orthogonal Ensemble of random matrix theory.

1 Introduction

A regular graph is a graph where each vertex has exactly the same degree, i.e. the same number of adjacent vertices. For a d-regular graph G, let $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$ be the eigenvalues of the adjacency matrix (defined in section 2). It can be easily shown that $\eta_1 = d$ and $|\eta_i| \leq d$ for the remaining eigenvalues. Let $\rho(G) = \max(|\eta_2|, |\eta_n|)$ be the second-largest eigenvalue of the graph G. Define the edge expansion constant $h(G)$ to be

$$h(G) = \min_{0 \leq |A| \leq n/2} \frac{|\partial A|}{|A|}$$

where A is any nonempty subset having at most $n/2$ vertices, and ∂A is the set of edges with exactly one endpoint in A. The edge expansion comes from the theory of expander graphs, where one wishes to construct an efficient network with a good connection property. Networks can be seen as vertices sharing data via edges connecting them. The edge expansion constant measures how well connected is the network by restricting the number of wires used in the network, but at the same time ensuring that any subset A is well connected to its complement $\bar{A}$. In the case G is a d-regular graph, the edge expansion constant is related to $\rho(G)$ by the following bounds derived by Dodziuk, Alon-Milman and Alon [Alo86, AM85, Dod84]:

$$\frac{d - \rho(G)}{2} \leq h(G) \leq \sqrt{2d(d - \rho(G))}$$

The quantity $d - \rho(G)$ is defined as the spectral gap of the graph G. If follows that the edge expansion of G is particularly significant when ρ is small. Moreover, a theorem derived by Alon and Boppana [Alo86, Nil91] states that $\rho(G) \geq 2\sqrt{(d-1)} - o_n(1)$. In the optimal case where $\rho(G) \leq 2\sqrt{d-1}$, the graph G is said to be Ramanujan. An open question is to prove the existence of infinite families of d-regular Ramanujan graphs for all $d \geq 3$ [ABG10], which would provide an infinite family of optimal expanders. The idea of random lifts was introduced by Friedman [Fri03] in order to obtain new Ramanujan graphs from smaller ones (namely base graphs). Addario-Berry and Griffiths showed [ABG10] that with extremely high probability, all eigenvalues of the random lift that are not eigenvalues of the base graph have order $O(\sqrt{d})$. This implies that if the base graph is Ramanujan, then the random lift is with high probability nearly Ramanujan.

In the original paper introducing random lifts [ALMR01], a variety of properties of random lifts are discussed: connectivity, expansion, independent sets, colouring and perfect matchings. In the present article, we first study the asymptotic eigenvalue distribution of random n-lifts as $n \to \infty$. We show that it follows the asymptotic law given in a paper by McKay [McK81]:

$$f(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} & \text{for } |x| \leq 2\sqrt{d-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The main part of the proof is to show that the number of short cycles of a fixed length k (denoted Z_k) tends to a constant, denoted λ_k, which depends only on the base graph G. By following the method of moments described by Janson, Luczak and Rucinsky [JLK00], we show precisely that Z_k is Poisson distributed according to the parameter λ_k.

The second part of our article is about the level spacing distribution of random lifts. First, define the sequence of unfolded eigenvalues to be $x_i = \{F(\eta_i)\}$, where F is the cumulative function associated with the asymptotic density (1.1). Then the quantities $s_i = x_{i+1} - x_i$ are called the spacings of the graph G. By following the work of Jakobson, Miller, Rivin and Rudnick [JMRR99], we give experimental results about the level spacing distribution of random lifts, and we show that there is a good fit between the random lift spacings and the Gaussian Orthogonal Ensemble (GOE) spacings. The GOE comes from the theory of random matrices; it is claimed that eigenvalues of large random symmetric matrices model the fluctuations of energy levels of chaotic dynamical systems [BGS84].

2 Short cycles in random lifts

2.1 Graphs and random lifts

We denote the set of vertices and the set of edges of graph G by $V(G)$ and $E(G)$ respectively. By definition, two vertices v_1 and v_2 are adjacent (or neighbors) if they are connected by an edge $e \in E(G)$ and the degree of a vertex v, denoted $\deg(v)$, is the number of edges adjacent to v. For the rest of the paper, we consider only simple graphs G such that all vertices have a fixed degree d. We call such graphs *d-regular simple graphs*. Note that simple graphs are graphs that contain no loops or parallel edges. An important tool in the study of graphs is the adjacency matrix $A(G)$ which is an $n \times n$ matrix, with $n = |V(G)|$, where a_{ij} is the number of edges from v_i to v_j. In the case of simple d-regular graphs, all diagonal entries of A are zero and the remaining entries are either 0 or 1. Moreover, the sum of the entries of each row and each column is equal to d.

We define a k-cycle to be a connected 2-regular subgraph whose edges and vertices are only traversed once, therefore containing k vertices and k edges. A closed walk of length k is defined as a sequence of adjacent vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ so that the first and last vertices are the same, i.e $v_1 = v_{k+1}$. A closed non-backtracking walk is defined as a closed walk such that for any vertex $v_i \in \{v_3, \ldots, v_{k+1}\}$, we have $v_i \neq v_{i=2}$. Conversely, a closed backtracking walk is a closed walk such that $v_i = v_{i-2}$ for at least one $i \in \{2, \ldots k+1\}$.

A *random n-lift* $L_n(G)$ of a graph G is obtained by replacing each vertex v_i of G by a set V_i of n vertices (called the fibre of v_i), and placing a random matching between V_i and V_j for each edge $(v_i, v_j) \in G$. We call G the *base graph* of the lift.

2.2 Short cycles in random lifts

Let $L_n(G)$ be a random n-lift of the d-regular graph G. Define Z_k as the number of cycles of length k in $L_n(G)$ for $k \geq 3$. Define c_k as the number of closed non-backtracking walks of length k in G. We first show that the expected number of k-cycles for a fixed k approaches a constant which depends only on the base graph as $n \to \infty$:

Lemma 2.1. $\mathbb{E}(Z_k) \to \frac{c_k}{2k}$ as $n \to \infty$.

Proof. Let p_k be the probability that a subset of k edges occurs in the lift $L_n(G)$. We will show that $p_k \sim \frac{1}{n^k}$ as $n \to \infty$, i.e. $\lim_{n\to\infty} \frac{p_k}{1/n^k} = 1$. Let Γ_k be the subgraph formed by the k edges and let $\pi(\Gamma_k)$ be the projection of Γ_k on the base graph G. For an edge $e \in \pi(\Gamma_k)$, let $m(e)$ be the number of edges in Γ_k projected on e. We call $m(e)$ the *multiplicity of the edge* e. Let s_i be the number of edges $e \in \pi(\Gamma_k)$ such that $m(e) \geq i$ and let r be the greatest multiplicity of an edge in $\pi(\Gamma_k)$. Then we have

$$s_r = k - \sum_{i=1}^{r-1} s_i$$

For an edge $e \in \pi(\Gamma_k)$ with multiplicity m, we have m corresponding edges in the subgraph Γ_k within the same fibre. This set of edges occurs with probability $\frac{1}{n}\frac{1}{n-1}\cdots\frac{1}{n-(m-1)}$. It follows that

$$\begin{aligned} p_k &= \prod_{i=1}^{r}\left(\frac{1}{n+1-i}\right)^{s_i} \\ &= \left(\prod_{i=1}^{r-1}\left(\frac{1}{n+1-i}\right)^{s_i}\right)\left(\frac{1}{n+1-r}\right)^{k-\sum_{i=1}^{r-1} s_i} \\ &= \left(\prod_{i=1}^{r-1}\left(\frac{n+1-r}{n+1-i}\right)^{s_i}\right)\left(\frac{1}{n+1-r}\right)^{k} \\ &\sim \frac{1}{n^k} \text{ as } n \to \infty \end{aligned}$$

Let w_k be a cycle of length k in $L_n(G)$. The projection $\pi(w_k) \in G$ must be a closed non-backtracking walk in the base graph G. Fix a closed non-backtracking walk $w'_k \in G$. Let $a(w'_k)$ be the number of possible cycles $w_k \in L_G(n)$ where $\pi(w_k) = w'_k$. We will show that $a(w'_k) \sim \frac{n^k}{2k}$ as $n \to \infty$.

Let t_i be the number of vertices which appear at least i times in w'_k, for $i = 1, 2, \ldots, l$ where l is the maximal occurrence of a vertex in w'_k. We have

$$t_l = k - \sum_{i=1}^{l-1} t_i$$

To count the number of possible cycles $w_k \in L_G(n)$, we count how many ways we can choose the vertices in w_k such that their projections are the vertices of w'_k. For a vertex u_j which appears for the first time in the walk, we have n choices for choosing a vertex of w_k in the fibre V_{u_j}. When a vertex u_j appears for its i-time, we have $(n-i+1)$ choices in V_{u_j} since $(i-1)$ vertices have already been chosen in this fibre. Since there are $2k$ possible ways to start the process, we have

$$
\begin{aligned}
2ka(w'_k) &= \prod_{i=1}^{l}(n+1-i)^{t_i} \\
&= \left(\prod_{i=1}^{l-1}(n+1-i)^{t_i}\right)(n+1-l)^{k-\sum_{i=1}^{l-1} t_i} \\
&= \left(\prod_{i=1}^{l-1}\left(\frac{n+1-i}{n+1-l}\right)^{t_i}\right)(n+1-l)^{k} \\
&\sim n^k \text{ as } n \to \infty
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbb{E}(Z_k) &= \sum_{w'_k} a(w'_k)p_k \\
&\sim \sum_{w'_k} \frac{n^k}{2k}\frac{1}{n^k} \text{ as } n \to \infty \\
&= \frac{c_k}{2k}
\end{aligned}
$$

which proves the lemma. □

Following the same idea for a general subgraph H of $L_n(G)$, one has the following lemma:

Lemma 2.2. Let H be a subgraph of $L_n(G)$ with v vertices and e edges. Then the expected number of such subgraphs in $L_n(G)$ is $O(n^{v-e})$. In the case $e > v$, we have $\mathbb{E}(H) = O\left(\frac{1}{n}\right)$.

Proof. The idea is similar to that of the proof of Lemma 1. We have already shown that the probability that a subset of e edges occurs in the random lift is $p^e \sim n^{-e}$. Moreover, as n increases, the number of possible vertices that can be chosen among a fibre, for a fixed vertex of the projection $\pi(H)$ on G, is of order n, i.e. $a(\pi(H)) = O(n^v)$ for any subgraph H containing v vertices. One concludes by noticing that taking the sum over the finite number of subgraphs H' such that $\pi(H) = H'$ does not affect the result for the asymptotic result, i.e.

$$\sum_{H' \in G : \pi(H') = H} \frac{1}{n^e} O(n^v) = O(n^{v-e})$$

□

Janson, Luczak and Rucinsky showed in [JLK00] that for random regular graphs, the number of cycles of length k is distributed according to a Poisson distribution with parameter $\theta_k = \frac{1}{2k}(d-1)^k$. They used the method of moments in the case of Poisson distributions and used specifically the following theorem:

Theorem 2.1 (Theorem 6.10 of [JLK00]). Let $(X_n^{(1)}, \dots, X_n^{(m)})$ be vectors of non-negative and bounded random variables, where $m \geq 1$ is fixed. If $\lambda_1, \dots, \lambda_m \geq 0$ are such that, as $n \to \infty$,

$$\mathbb{E}((X_n^{(1)})_{k_1} \dots (X_n^{(m)})_{k_m}) \to \lambda_1^{k_1} \dots \lambda_m^{k_m}$$

for every $k_1, \dots, k_m \geq 0$, where $\mathbb{E}(X)_i$ denotes the i-th factorial moment of X, then $(X_n^{(1)}, \dots, X_n^{(m)}) \to^d (Z_1, \dots, Z_m)$, where $Z_i \in Po(\lambda_i)$ are independent Poisson variables.

We will show a similar result for random n-lifts:

Theorem 2.2. Let $\lambda_k := \frac{c_k}{2k}$, where c_k is the number of closed non-backtracking walks in the base graph G, and let $Z_{k\infty} \in \text{Poisson}(\lambda_k)$ be independent Poisson distributed random variables, $k = 1, 2, 3, \ldots$ Then the random variables $Z_k(L_n(G))$ converge in distribution to $Z_{k\infty}$, i.e. $Z_k(L_n(G)) \to^d Z_{k\infty}$ as $n \to \infty$, jointly for all k.

Proof. We need to compute the factorial moments $\mathbb{E}(Z_k)_i$ for all $i \geq 2$. We begin with $\mathbb{E}(Z_k)_2$. By definition $\mathbb{E}(Z_k)_2 = \mathbb{E}(Z_k(Z_k - 1))$, i.e. $\mathbb{E}(Z_k)_2$ is the expected number of pairs of two distinct cycles of length k. We write $\mathbb{E}(Z_k)_2 = Y' + Y''$ where Y' is the number of pairs of vertex disjoint cycles and Y'' is the number of pairs of two cycles with at least one common vertex. We can decompose Y'' further according to the number of common vertices and to the number of total edges in the pair. Then $Y'' = \sum_{j=1}^{J} Y_j''$ where J depends only on k. Since the Y_j'' count the number of some subgraphs which have more edges than vertices, we have $\mathbb{E}(Y_j'') = O\left(\frac{1}{n}\right)$ for all j by Lemma 2. It follows that $\mathbb{E}(Y'') = O\left(\frac{1}{n}\right)$ since J does not depend on n. It remains to show that $\mathbb{E}(Y') \to \lambda_k^2$. We proceed in a similar way as we did for $\mathbb{E}(Z_k)$ in Lemma 1: We have $p_{2k} \sim \frac{1}{n^{2k}}$. Let $w_{kk} = w_k^{(1)} \sqcup w_k^{(2)}$ be a pair of two disjoint k-cycles in $L_n(G)$. The projected walks $\pi(w_k^{(1)})$ and $\pi(w_k^{(2)})$ are two closed non-backtracking k-walks in G which may intersect or not. Moreover, it is possible that $\pi(w_k^{(1)}) = \pi(w_k^{(2)})$. Let $w'_{kk} = \pi(w_k^{(1)}) \sqcup \pi(w_k^{(2)})$. For any $w'_{kk} \in G$, let $d(w'_{kk})$ be the number of pairs of two disjoint k-cycles $w_{kk} \in L_n(G)$ such that $w'_{kk} = \pi(w_{kk})$. As $n \to \infty$, we have $d(w'_{kk}) \sim \left(\frac{n^k}{2k}\right)^2$. Summing over all possible pairs $w'_{kk} \in G$, we get $\sum_{w'_{kk}} d(w'_{kk}) \sim \left(\frac{c_k n^k}{2k}\right)^2$. If follows that $\mathbb{E}(Y') \sim (\lambda_k)^2$ and $\mathbb{E}(Z_k)_2 \sim (\lambda_k)^2$. The same argument applies on any factorial moment $\mathbb{E}(Z_k)_i$ and for any combination $\mathbb{E}((Z_k)_{k_1} \ldots (Z_k)_{k_m})$. By Theorem 1, the proof is complete. □

To illustrate this result, we give as an example the simple case where $G = K_{d+1}$, the complete d-regular graph on $(d+1)$ vertices.

To count the number of non-backtracking walks of length k, we use the following idea. First, we choose the first vertex of the walk, which gives $d+1$ possibilities. For the second vertex, we have d possibilities. For the third vertex, since backtracking is not allowed, we are left with only $(d-1)$ possibilities. In general, for the i-th vertex, with $2 \leq i \leq k-2$, we have $(d-1)$ possibilities as well. For $i = k-1$, we cannot choose the initial vertex of the walk neither, since this would imply backtracking at the end of the walk; we therefore have $(d-3)$ choices for the $(k-1)$-th vertex. It follows that

$$\lambda_k = \frac{c_k}{2k} = \frac{(d+1)d(d-1)^{k-3}(d-3)}{2k}$$

Let A be the adjacency matrix of the graph $L_n(G)$. It is easy to see that $\text{tr}(A^k)$ gives the total number of closed walks of length k in the graph $L_n(G)$. We will analyze the case $k = 4$. To compute $Z_4(L_n(G))$ from the adjacency matrix, we need a relation between $Z_4(L_n(G))$ and $\text{tr}(A^4)$. To do so, we need to subtract from $\text{tr}(A^4)$ the number of closed walks which are not cycles. Fix a vertex $v \in L_n(G)$. There are two types of backtracking closed walks of length 4, presented in Figure 1.

For the first type, there are d^2 such walks since there are d ways of choosing v_1 and d ways of choosing v_2. For the second type, there are $d(d-1)$ such walks. Since there are $(d+1)n$ vertices in $L_n(G)$, we have the following relation:

$$Z_4 = \frac{tr(A^4) - nd(d+1)(2d-1)}{8}$$

We divided the right-hand side by 8 since every walk of length 4 is being counted exactly 8 times: 4 ways to fix a vertex and 2 ways to traverse the cycle. We computed Z_4 for 500 random lifts of the complete graph K_9 with $n = 200$ and compared the empirical distribution with the asymptotic Poisson distribution of Z_4. By using the formula above, we get λ_4 to be 378 $(9 \times 7 \times 6)$. By looking at the density of the Poisson distribution with $\lambda = 378$ against the empirical distribution (see Figure 2), we observe that the expected behavior of Z_4 for large n is a very good fit for n as small as 200.

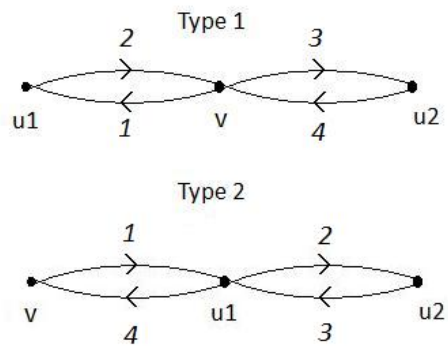

Figure 2.1: Backtracking closed walks of length 4 starting at the vertex v. Numbers represent the steps of the walk.

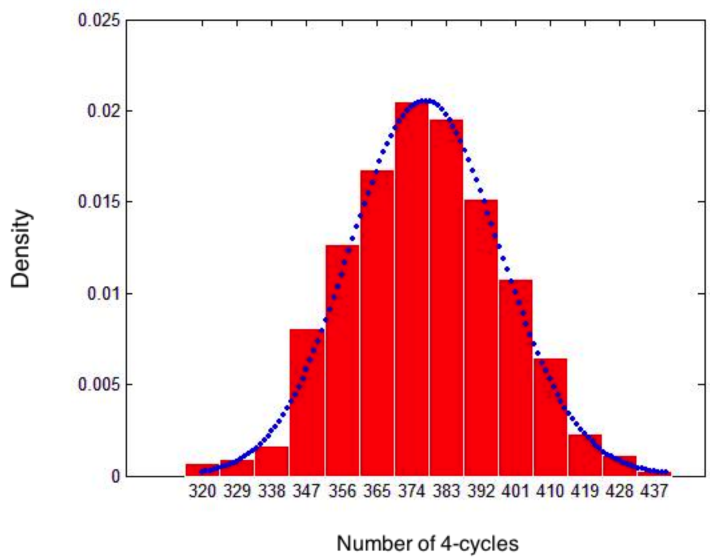

Figure 2.2: Number of 4-cycles for 500 n-lifts of the complete graph K_9 with $n = 200$ vs. the theoretical Poisson distribution of Z_4.

3 Asymptotic eigenvalue distribution of random lifts

We will study here the eigenvalues of $L_n(G)$ where G is a d-regular graph and when $n \to \infty$. We notice first that the eigenvalues of the base graph G are all inherited by $L_n(G)$. To see this, let η be an eigenvalue of $A(G)$ with eigenvector x. Define the vector y in the following way: $y_v = x_i$ if $v \in V_i$ and where V_i is the fibre of the vertex $u_i \in G$. Then y is an eigenvector or $L_n(G)$ with eigenvalue η. In McKay [McK81], the following theorem is proved:

Theorem 3.1. Let $X_1, X_2, \ldots$ be a sequence of regular graphs, each of degree $v \geq 2$, which satisfies the conditions

- $n(X_i) \to \infty$ as $i \to \infty$ where $n(X_i)$ is the number of vertices of the graph X_i
- for each $k \geq 3$, $\frac{Z_k(X_i)}{n(X_i)} \to 0$ as $i \to \infty$

Let $f(X_i, x)$ be the density distribution of the eigenvalues of $A(X_i)$. Then for each $x, f(X_i, x) \to f(x)$ as $i \to \infty$, where $f(x)$ is the function defined as follows:

$$f(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} & \text{for } |x| \leq 2\sqrt{d-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We refer to $f(x)$ as *McKay's law.* Since we proved previously that $\mathbb{E}(Z_k) \to \lambda_k(G)$ as $n \to \infty$, we have

$$\mathbb{E}\left(\frac{Z_k}{|L_n(G)|}\right) \sim \frac{\lambda_k(G)}{nd} \to 0 \text{ as } n \to \infty,$$

which shows that the second condition of the theorem is satisfied. We conclude that the asymptotic eigenvalue distribution of random lifts follows McKay's law.

4 Experimental results

We computed the empirical eigenvalue distributions of two ensembles of graphs obtained by lifting two different base graphs: the first is a ensemble of 200-lifts of the complete graph K_5 (see Figure 3A) and the second is a ensemble of 200-lifts of a random generated 3-regular graph with 6 vertices (see Figure 3B). In both figures, the empirical distributions are compared to McKay's law. The reader has to notice that we did not include the old eigenvalues of the lifts (the eigenvalues inherited from the base graph) in the distributions.

In [JMRR99], it is conjectured that the level spacing distribution of random regular graphs is similar to that of the Gaussian Orthogonal Ensemble (GOE), which is a statistical model in Random Matrix Theory. The empirical level spacing distribution is obtained in the following way: We first unfold the spectrum by setting

$$x_j = F(\eta_j)$$

where $F(x)$ is the cumulative distribution function associated to McKay's law. Then the sequence of numbers $\{x_j\}$ has unity as mean spacing. We consider the spacings $s_n = x_{n+1} - x_n$. The distribution function of the s_n is called the *level spacing distribution.* For the GOE, an approximation derived by Wigner is known for the level spacing distribution (called *Wigner surmise*):

$$P_W(s) = \frac{\pi}{2} s e^{\frac{-\pi s^2}{4}}$$

We computed the level spacing distribution of our previous graphs and we plotted the results in comparison to the Wigner surmise (Figure 4). The results show a good fit and lead us to think that the eigenvalues of random lifts have GOE spacings.

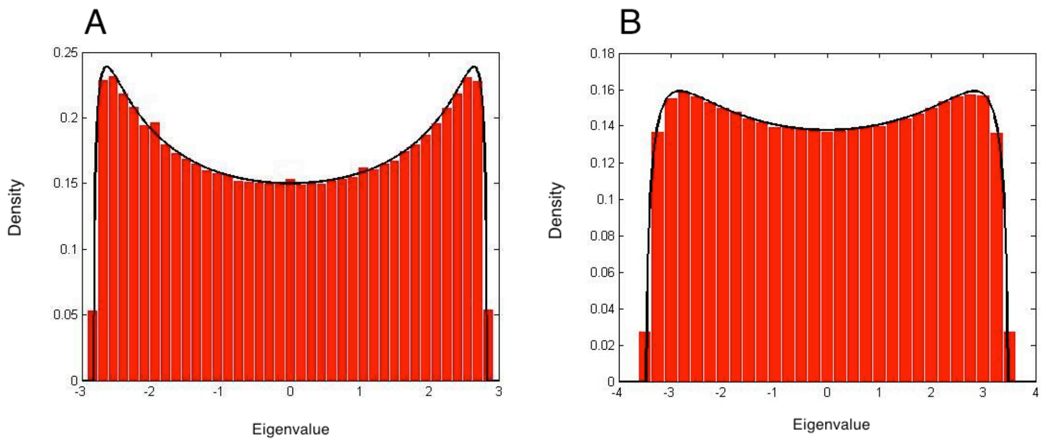

Figure 4.1: A) Eigenvalue distribution of 200-lifts of the complete graph K_5 vs McKay's law B) Eigenvalue distribution of 200-lifts of a random 3-regular graph on 6 vertices vs McKay's law

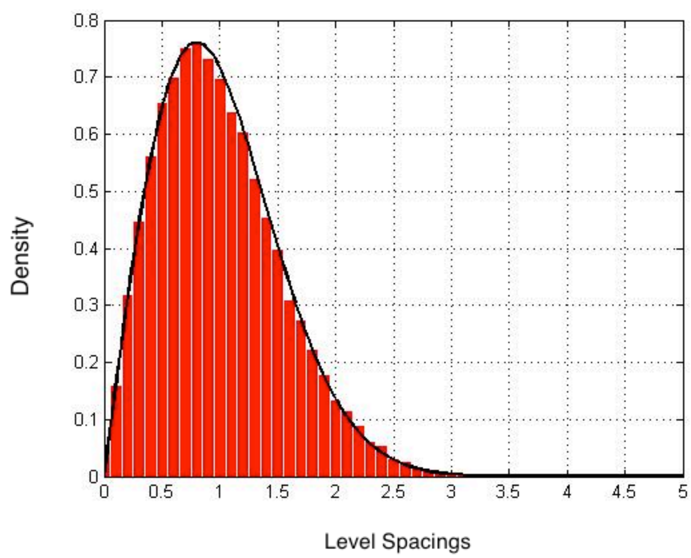

Figure 4.2: Level spacing distribution of 100-lifts of a random 3-regular graph on 6 vertices *vs* GOE Wigner surmise

5 Random lift generation

For the experiments, we used three different base graphs: complete graphs K_{d+1}, complete bipartite graphs $K_{d,d}$ and random d-regular graphs. For generating random d-regular graphs, we used the configuration model (Bollobas) described in [JLK00]. Briefly, an $2d$-configuration of a graph G is a partition of the cartesian product $W = \{1, \dots, |G|\} \times \{1, \dots 2d\}$ into dn pairs, where $|G| = n$ is the number of vertices of the graph G. The natural projection of the configuration W onto G creates a d-regular graph G'.

Now, to construct a random n-lift H from a base graph G, construct first the $nv \times nv$ matrix A which represents the adjacency matrix of H, where $v = |V(G)|$. The v first columns (or rows) represent the vertices of the first copy of G. The next v columns represent the vertices of the second copy of G, and so on. For each edge $(i, j) \in G$, we construct an array X of size n. For the k-th entry of X, we generate a random number Rand(k) between 0 and m where m is much larger than n. We sort the array X in such a way that we keep track of the original indices. Denote the new index of Rand(k) by $\pi(k)$. The associations k to $\pi(k)$ create a perfect matching between the fibres V_i and V_j. For each pair $(k, \pi(k))$, we set

$$A_{(i+kn,j+\pi(k)n)} = 1 \text{ and } A_{(j+\pi(k)n,i+kn)} = 1$$

6 Acknowledgements

We would like to thank Dr. Louigi Addario-Berry and Dr. Dmitry Jakobson for introducing us to the project, and the Institut des sciences mathématiques (ISM) for the Undergraduate Summer Scholarship awarded to J.-P. Fortin under the supervision of Dr. Nikolay Dimitrov.

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