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THE COVER: The cover shows the Petersen graph, superimposed with several of its random lifts of different degrees. Connecting the various probability distributions associated with random lifts to the base graph form a part of the modern probabilistic method in combinatorics. Some of these ideas are explored in the article by Jean-Philippe Fortin and Samantha Rudinski in this issue. The cover was designed by Edgar A. Bering IV using a combination of custom software and Adobe Illustrator.

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Remarks

From the Editors

Dear Reader,

The Waterloo Math Review would like to thank Eeshan Wagh and Frank Ban for their service as editors over the years. They were there for the inception of the journal and their hard work has been pivotal in making the journal a success. We would like to wish them the best of luck and success in graduate school. We would also like to welcome Michael Baker and Ehsaan Hossain to the editorial staff.

In this issue, we decided to combine the fall and winter issues in order to improve the quality of the journal. We are very pleased with the selection of papers in this issue as they cover a variety of subjects including applied math, combinatorics and pure math. We are always pleased to see papers from across Canada as it shows the quality of research Canadian undergraduate students are capable of producing.

We have seen tremendous growth this past year attributed to both an increase in submissions and readership. As the Waterloo Math Review becomes more established and experienced, we hope to continue expanding our readership. We are putting the infrastructure in place to ensure the continued success of the Waterloo Math Review in the upcoming years.

Regards,
Michael Baker
Frank Ban
Ehsaan Hossain
Saifuddin Syed
Eeshan Wagh
Editors
editor@mathreview.uwaterloo.ca

From the CUMC

CUMC is a conference held every year, in a different Canadian university each time. The CUMC, which is supported by the Canadian Mathematical Society through it's Student Committee is held this year at Universit de Montral from July 10th to July 14th. The conference offers a unique opportunity for students to acquire a first exposure to current research in different mathematical fields. Through seminars and social activities, the CUMC also provides venue for students to connect with other students from across Canada connections students can benefit from throughout their mathematical career. Furthermore, the event seeks to promote different fields within mathematics, and offers a unique opportunity for students to meet active researchers working in applied fields, such as physics, economics, informatics, statistics, engineering, and actuarial science.

The conference brings together more than two hundred students from various academic backgrounds. Each student has the opportunity to present a topic they find interesting. It is a rewarding experience for talented young mathematicians to go beyond the typical undergraduate experience and meet to discuss their common interests with likeminded students. It is a non-competitive event because we believe that such a context best facilitates the exchange of ideas and encourages diversity, which is essential for the development of mathematics across the country.

One of the principal perks of CUMC is allowing students to present their own work or any interesting mathematics theorem in a 20 or 45 minute presentation. This gives anyone who wants to give such a presentation an opportunity to learn how to present mathematics, and how to speak in front of a crowd. The website, http://cumc.math.ca is the main source of information regarding the event. If you do not find what you are looking for, please do not hesitate to write to cumc@cumc.math.ca. It will be our pleasure to answer your questions.

A brief introduction to measurable cardinals

Edgar A. Bering IV
University of Illinois at Chicago
eberin2@uic.edu

Abstract: We introduce the notion of a measurable cardinal, motivated by examples from measure theory. We then develop some initial inaccessibility results for such cardinals and summarize Solovay's results regarding the consistency of set theory with the hypothesis that there is an accessible measurable cardinal. These results lead us to a question that appears to be open (it is certainly open ended) regarding the consistency of weaker forms of choice and the existence of measurable cardinals. This paper aims to be self contained and accessible to an advanced undergraduate; however, its motivation rests in measure theory so previous exposure to the subject will be helpful.

1 Introduction

Measure theory is the study of assigning size to subsets of a given set (the assignments are known as measures). Measures are required to interact sensibly with set operations, though a large portion of measure theory is concerned with measures compatible with other structures on a space. Note that injections commute with set operations. Therefore, when a measure is defined on every subset of a set we can obtain a measure on any larger cardinality. In light of this, in the search for measures with interesting properties, we can restrict ourself to the smallest cardinality admitting a measure with a given property.

Counting measure (the size of a set is its cardinality) and Dirac measures (a subset is assigned size infinity if it contains a given element and size zero otherwise) assign a size to every subset, but they are uninteresting in most applications of measure theory. Interesting examples of measures (Lebesgue measure on $\mathbb{R}^n$, the natural probability measure on $\{0,1\}^{\mathbb{N}}$, Radon measures on uncountable locally compact Hausdorff spaces) must exclude a class of problematic subsets to be defined. This leads us to ponder: what makes these measures 'interesting'? And is there a cardinality high enough to admit an interesting measure defined on the power set of a set of that cardinality? Since the smallest cardinal with an interesting measure will provide interesting measures on larger cardinalities we will give it particular attention.

Cardinals that admit such measures were first considered by Ulam [Ula30]. The central result regarding such cardinals affirms the belief that measure theory must make do with non-measurable sets and work around them. To be specific, it is known that if the commonly accepted Zermelo-Frankel set theory with the axiom of choice (ZFC) is consistent then we cannot prove, using ZFC, that *any* cardinality admits an interesting measure. Both this result and classical results regarding the existence of non-measurable sets rely on the Axiom of Choice; we are lead to suspect it is the source of the trouble.

Solovay explored this line of reasoning, and showed that there is a model of ZF (the Zermelo-Frankel axioms without choice) set theory where the axiom of choice does not hold and every subset of the real numbers is Lebesgue measurable [Sol70]. In the interest of making this paper accessible to readers not well versed in modern set theory we omit a detailed discussion of Solovay's results. Further results of Solovay show that ZF with a weaker variants of choice are consistent with interesting measures defined on the power set of familiar cardinals [Sol71].

This paper is organized as follows. Section 2 introduces the necessary background for this paper, both set theory and measure theory. The exposition is necessarily terse and proofs are omitted, but it serves to make the paper self contained. Readers already familiar with these topics will not be harmed by reading lightly here. Section 3 explores formally the interesting properties of a measure desired for our investigation. Section 4 develops the central result of the paper, that in ZFC we cannot hope to prove that any cardinality

admits an interesting measure. Section 5 discusses briefly the consistency of the existence of cardinalities possessing nice measures in ZF set theory with weaker versions of choice.

2 Background: Zermelo-Frankel set theory and basic measure theory

2.1 Formal systems, models, and incompleteness

In taking after Hilbert, if we regard mathematics as a formal game played symbolically with an alphabet we can then analyze this game mathematically. (A philosophically inclined reader may object to this particular portrayal. A detailed discussion of the philosophy at play here is very far afield indeed, so we acknowledge the objection, ignore it, and move on.) To this end we introduce the language of first order logic. The alphabet is $( ) \wedge \vee \rightarrow \leftrightarrow \neg = \forall \exists v_0 v_1 \ldots$. The list of variables is understood to be countable. When discussing a particular bit of mathematics we enrich this base alphabet with constant symbols (conventionally $c_0, c_1, \ldots$), relation symbols (denoted $R_0, R_1, \ldots$), and function symbols (denoted $f_0, f_1, \ldots$). Function and relation symbols are specified along with a fixed arity for each symbol. We restrict our attention to strings in this language that mirror those we use in mathematics regularly, the syntax mirrors standard notation (though brackets are mandatory), and these are known as *well-formed formulas*. Proofs are modeled by a set of rules for combining basic proof steps, known as a sequent calculus, these rules mirror logical deduction like one would expect (for example, from $(\phi \rightarrow \psi) \wedge \phi$ we may conclude ψ). If all variables in a well-formed formula (string with valid syntax) are preceded with a quantifier ($\forall$ or $\exists$) they are said to be *bound* and such a formula is known as a *statement*. Any variable not preceded by a quantifier is said to be *free*.

Definition 2.1. A *language* is the set of all formulas that can be formed using a fixed collection of constant, relation, and function symbols.

Definition 2.2. A *formal system* is some collection of statements Σ in a language. The statements in Σ are usually referred to as *axioms*.

Definition 2.3. A statement ϕ is *derivable* in a formal system Σ if from Σ we can create ϕ with finitely many applications of first order logic rules. Such a ϕ is a *theorem* of Σ, and we write $\Sigma \vdash \phi$.

Definition 2.4. A formal system Σ is *consistent* if there is no statement ϕ such that $\Sigma \vdash \phi \wedge \neg\phi$.

Definition 2.5. A *model* of a formal system M is a set along with assignments of the constant symbols to elements of M, the relation symbols to subsets of direct products of M, and the function symbols to functions from direct products of M to M such that every statement in Σ is true in M (the quantifiers are understood to range over M).

We note that if a system has a nonempty model then it must be consistent.

An important result in the theory of formal systems is Gödel's Second Incompleteness theorem. Using a numeric encoding of statements of a formal system (the binary of the ASCII values of the characters in the string will work, provided the encoding is unique), if a system Σ can encode and prove the axioms of standard arithmetic it can be self referential in the following fashion. If n^ϕ is the numeric encoding of ϕ we can define, in arithmetic, the statement $\mathrm{Der}_\Sigma(n^\phi)$ that is true if $\Sigma \vdash \phi$. Then, if in Σ we can prove the axioms of arithmetic, the statement $\neg\mathrm{Der}_\Sigma(n^{0=1})$, denoted Consis_Σ is a statement in the language of Σ. Gödel's theorem can then be stated:

Theorem 2.1 (Second Incompleteness). If a formal system Σ (with computable axioms) can encode and prove the axioms of standard arithmetic and Σ is consistent then it is not the case that $\Sigma \vdash \mathrm{Consis}_\Sigma$.

Our discussion here is necessarily lacking in formality and detail. The theory of formal systems is a field of study in its own right and tends to be very verbose to state in full detail. The interested reader is referred to any standard introduction to the subject, such as the book by Ebbinghaus, Flum, and Thomas [EFT94], for a detailed development of the theory.

2.2 THE AXIOMS OF ZERMELO-FRANKEL SET THEORY

We now introduce the formal system taken by most mathematicians as the formal foundation of mathematics, the system of Zermelo-Frankel set theory. We present the axioms both as a convenient reference and a brief introduction. We include an informal discussion of each axiom. The language used has no constant symbols or function symbols and a single relation symbol $\in$. For clarity we will use x, y, z, u, v, w for variables, it is understood that we can re-write this only in terms of v_i, the actual variable symbols.

Axiom 1 (Extensionality).

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

This axiom requires sets to be defined by their elements.

Axiom 2 (Empty Set).

$$\exists x \forall y(\neg y \in x).$$

This axiom specifies the existence of *an* empty set. We denote a set with this property $\emptyset$, but caution that $\emptyset$ is not a constant symbol in our language, and it may not be unique in all models.

Axiom 3 (Pair).

$$\forall x, y \exists z \forall w(w \in z \leftrightarrow w = x \vee w = y).$$

Typical notation for z is $\{x, y\}$, and $\{x\}$ for $\{x, x\}$. (Note that the axioms, as written do not give us a method for creating "the set containing x", and even if such an axiom were included it would be redundant, as we would find $\{x\} = \{x, x\}$ by Extensionality.)

Axiom 4 (Union).

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists t(z \in t \wedge t \in x)).$$

This axiom states that y is the union of all elements of x. Using Axiom 3 we can construct $z = x \cup y$.

Axiom 5 (Infinity).

$$\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x)).$$

We may construct the natural numbers by taking $0 = \emptyset$ and using the successor function $x \mapsto \{x, \{x\}\}$. This axiom then guarantees us a set of all natural numbers. This axiom is required to guarantee an infinite set; there are models of the other axioms where every set is finite. In light of this a natural question, that our discussion will touch on, is whether or not infinities beyond the reach of these axioms exist.

The next axiom is not a single axiom, but an axiom *schema*, a computable process for generating countably many axioms.

Axiom 6 (Replacement). If ϕ is a formula with at least two free variables:

$$\forall t_1, t_2, \ldots t_k(\forall x \exists! y \phi(x, y, t_1, \ldots, t_k) \rightarrow \forall u \exists v B(u, v)),$$

where $B(u, v)$ is the formula:

$$\forall r(r \in v \leftrightarrow \exists s(s \in u \wedge \phi(s, r))).$$

The notation $\exists!$ is for "there exists a unique" and can be formed in first order logic. The hypothesis of this axiom is that ϕ encodes a partial function, its conclusion is that the range of ϕ over sets is also a set. This is indeed a countable schema, there are countably many strings over our alphabet, and only some subset of them are formulas with two free variables.

Axiom 7 (Power Set).

$$\forall x \exists y \forall z(z \in y \leftrightarrow \forall u(u \in z \rightarrow u \in x)).$$

This axiom guarantees a power set for every set. While it may appear that this is subsumed by replacement the formula not quantified does not code a function, consider $x = \emptyset$. Indeed, Power Set implies an uncountable set and there is a model of the other axioms in which every set is countable.

Axiom 8 (Regularity).

$$\forall x \exists y (x = \emptyset \vee (y \in x \wedge \forall z (z \in x \to \neg z \in y))).$$

This is a technical axiom that is not used directly in higher mathematics. It specifies that as a partial order, $\in$ is always well-founded; i.e. there is an element that is minimal with respect to $\in$. (This does not make $\in$ a well-order on every set.) This axiom serves to exclude $x \in x$ and other paradox inducing sets. The reader already familiar with set theory may protest that we have left out the axiom schema of separation. No such omission has been made, separation is a consequence of empty set and replacement [Dev93].

These eight axioms make up what we will refer to as system ZF. There is another axiom taken by most modern mathematicians, the Axiom of Choice.

Axiom 9 (Choice).

$$\forall x (x \in z \to \neg x = \emptyset \wedge \forall y (y \in z \to x \cap y = \emptyset \vee x = y)) \to \exists u \forall x \exists v (x \in z \to u \cap x = \{v\}).$$

We note that $x \cap y$ can be defined using Replacement; for a fixed x the set $x \cap y$ is the image of y under $\phi(y, z) = \forall u (u \in z \leftrightarrow u \in x \wedge u \in y)$. Informally, Choice reads "if x is a family of sets then there is a set u made of one of each element of the family x". Note that Choice allows us to make potentially uncountably many choices, even if we have no concrete property ϕ that we can appeal to Replacement with. We denote system ZF with the addition of Choice by ZFC.

Recalling our discussion of incompleteness, we remark that ZF can prove the axioms of standard arithmetic, and so cannot prove $\text{Consis}_{\text{ZF}}$; similarly ZFC cannot prove $\text{Consis}_{\text{ZFC}}$, assuming these systems are consistent. Since our definition of a model is in terms of sets (treated intuitively, not as parts of a formal system), we can use this coding scheme to represent informal sentences about models of ZFC as formal statements about *sets in* ZFC. We can also code and prove as a theorem, inside ZFC, that the existence of a nonempty model implies consistency. It then follows from the incompleteness theorem that ZFC cannot be consistent *and* prove that a model of ZFC exists.

2.3 Ordinal numbers

In set theoretic investigations the ordinal numbers provide a useful generalization of the order and induction properties of the natural numbers. In the interests of brevity we will simply state a few results about ordinals that will be useful later, the interested reader can check any standard set theory introduction [Dev93].

Definition 2.6. A *well-ordering* on a set A is a relation $<$ such that for all $x, y \in A$ exactly one of $x < y, y < x$, or $x = y$ is true, and if $x, y, z \in A$ and $x < z, z < y$ then $x < y$, and A has a least element with respect to this order.

Definition 2.7. A set is *transitive* if $\forall z \in A$ we have $y \in z \to y \in A$. That is, $z \subseteq A$.

Definition 2.8. An *ordinal* is a transitive set α well-ordered by $\in$.

We attempt to capture the natural hierarchy of well-ordering structures with this definition. One quickly sees that $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ are all ordinals (we denote this sequence $0, 1, 2, \ldots$ and take it to be the definition of the natural numbers) and that in general if α is an ordinal then $\alpha \cup \{\alpha\}$ is an ordinal.

Theorem 2.2. An ordinal is the set of all ordinals that precede it.

Remark 2.3: This lets us order the ordinals by $\in$, and it well-orders them.

Theorem 2.4. If α is an ordinal then $\beta = \alpha \cup \{\alpha\}$ is the least ordinal greater than α, we write $\beta = \alpha + 1$.

Theorem 2.5. If S is a set of ordinals then there is a least ordinal in S.

Theorem 2.6. If S is a set of ordinals then there is a least ordinal α such that $\beta \in S \to \beta < \alpha$. We write $\alpha = \sup S$.

Definition 2.9. α is a *successor* ordinal if there is an ordinal β such that $\alpha = \beta + 1$. α is a *limit ordinal*] otherwise.

Theorem 2.7. There exists a limit ordinal. The least limit ordinal is the set of the natural numbers, denoted ω.

The following theorem generalizes induction and recursion to higher infinities.

Theorem 2.8 (Transfinite Induction). Let $P(\alpha)$ be a property defined on the ordinals. Suppose that $P(\emptyset)$ is true and $P(\alpha)$ holding for all $\alpha < \beta$ implies $P(\beta)$. Then P is true for all ordinals.

Note that we often treat the cases of limit and successor ordinals separately when conducting proof by transfinite induction.

2.4 CARDINAL NUMBERS

Cardinal numbers are used to discuss the relative size of sets. The notions of injective, surjective, and bijective function can be defined in ZF, so the following discussion makes sense in ZF. If there is a bijection between two sets we say they are similar and we denote this $x \sim y$. We also say that $x \preceq y$ if $x \sim z$ for some $z \subseteq y$. We note that in ZF we can show that $x \sim y$ is an equivalence relation and that if $z \subseteq x \subseteq y$, $z \preceq x \preceq y$.

Theorem 2.9 (Cantor-Schröder-Bernstein). $\text{ZF} \vdash (x \preceq y \wedge y \preceq x) \leftrightarrow x \sim y$, or in English, if there are injections $x \to y$ and $y \to x$ then x is similar to y.

Definition 2.10. An ordinal α is an *initial ordinal* if for all $\beta < \alpha$, $\neg(\beta \sim \alpha)$, that is, it is not similar to any earlier ordinal.

Definition 2.11. $\aleph(x) = \{\alpha | \alpha \preceq x\}$. The aleph function collects all ordinals similar to a subset of its argument.

Theorem 2.10. For any ordinal α, $\aleph(\alpha)$ is an ordinal, and for an initial ordinal $\aleph(\alpha)$ is the next initial ordinal.

Since initial ordinals create new levels in the order $\preceq$ we define:

Definition 2.12. A *cardinal* is an initial ordinal.

Using the $\aleph$ function and transfinite induction we can define a hierarchy of increasingly large cardinals

$$\begin{aligned} \aleph_0 &= \omega \\ \aleph_{\alpha+1} &= \aleph(\aleph_\alpha) \\ \aleph_\beta &= \bigcup_{\alpha<\beta} \aleph_\alpha \text{ for limit ordinals } \beta. \end{aligned}$$

In addition to the aleph function we can use a theorem of Cantor to get larger cardinals.

Theorem 2.11. In ZF $\neg(\mathcal{P}(x) \preceq x)$.

In words, the power set of a cardinal is a set of larger cardinality. By analogy with finite sets, for a cardinal κ we write 2^κ for the cardinal similar to $\mathcal{P}(\kappa)$.

For an arbitrary set let $|x| = \cap\{\alpha \,|\, \alpha \sim x,\ \alpha \text{ is an ordinal}\}$. If x is not similar to any ordinal then $|x| = \emptyset$, thought this cannot happen in ZFC—in ZFC every set is well-order-able and therefore similar to some ordinal. It is easy to check that $|x|$ is an initial ordinal, that for initial ordinals $|\alpha| = \alpha$, and that if $|x| \neq \emptyset$ and $x \sim y$ then $|x| = |y|$. We call $|x|$ the *cardinality* of x.

We can define a notion of cardinal addition; for an index set I we define $\sum_{i\in I} \kappa_i = \lambda$ by

$$\sum_{i\in I} \kappa_i = \left| \coprod_{i\in I} \kappa_i \right|,$$

where $\amalg$ stands for disjoint union, though we cannot show that this union is nonempty without the Axiom of Choice. We also remark that $\preceq$ is not necessarily a total order on sets in the absence of Choice.

With the notion of cardinal defined we now introduce a class of peculiarly large cardinals that will play a role later in the paper.

Definition 2.13. A cardinal κ is *inaccessible* if:

1. If $\lambda \prec \kappa$, κ cannot be written $\Sigma_{i\in\lambda}\theta_i$ for $\theta_i \prec \kappa$; that is, if κ is a cardinal sum, then either one summand is already of cardinality κ or there are at least κ summands
2. If $\lambda \prec \kappa$ then $2^\lambda \prec \kappa$.

ω is inaccessible, and in fact is a model of the axioms of ZFC other than Infinity. Since larger inaccessibles contain ω, similar reasoning shows that these higher cardinals are models of ZF and ZFC. (An example of the aforementioned reasoning, the second condition implies that for any proper subset $A \subset \kappa$ something behaving like the power set $\mathcal{P}(A)$ can also be found as a subset of κ, so the Power Set axiom is satisfied.)

2.5 Measure theory

We now briefly recall the relevant definitions and results form measure theory for the unfamiliar reader. While this paper aims to be accessible with no further measure theory, the examples of measure spaces given in a standard reference, such as Halmos's [Hal74], are occasionally referred to and will help the reader put the discussion in context.

Let X be a set.

Definition 2.14. We say a family of sets $\mathcal{M} \subseteq \mathcal{P}(X)$ is a *σ-algebra* (over X, if not clear from context) if

1. $\emptyset \in \mathcal{M}$.
2. $E \in \mathcal{M}$ implies $X \setminus E \in \mathcal{M}$.
3. If $E_1, E_2, \ldots \in \mathcal{M}$, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

For any set X, $\mathcal{P}(X)$ is a σ-algebra.

Definition 2.15. A function $\mu : \mathcal{M} \to [0, \infty]$ defined on a σ-algebra $\mathcal{M}$ is a *measure* if

1. $\mu(\emptyset) = 0$.
2. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ when $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. This property is known as *σ-additivity.*

Definition 2.16. The triple $(X, \mathcal{M}, \mu)$ for a set X, σ-algebra $\mathcal{M}$, and measure μ on $\mathcal{M}$ is a *measure space on* X.

Example 2.1. Let $a \in X$ be an arbitrary element. Then $(X, \mathcal{P}(X), \delta_a)$ where

$$\delta_a(E) = \begin{cases} \infty & \text{if } a \in E \\ 0 & \text{otherwise.} \end{cases}$$

is a measure space on X. δ_a is known as a *Dirac measure* on X.

Example 2.2. It is easily seen that for $(X, \mathcal{P}(X), \gamma)$ where

$$\gamma(E) = \begin{cases} n & \text{if } |E| = n \\ \infty & \text{if } \omega \preceq E \end{cases}$$

is a measure space on X. γ is known as the *counting measure* on X.

Example 2.3. Let $\mathcal{M}$ be the collection of countable and co-countable (those sets $E \subset X$ such that $X \setminus E$ is countable) subsets of X. Then $\mathcal{M}$ is a σ-algebra and $(X, \mathcal{M}, \mu)$ where $\mu(E) = 1$ if E is co-countable and zero otherwise is a measure space.

Example 2.4. Let $(\Omega, \mathcal{B}, P)$ be a sample space. Probability theorists call σ-algebras *σ-fields*, and it is readily seen that the probability of some event (subset of Ω in the σ-field) is a measure.

The following theorem summarizes the basic properties of measures that will be used in this paper.

Theorem 2.12. If $(X, \mathcal{M}, \mu)$ is a measure space, then:

1. If $E, F \in \mathcal{M}$, $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $E_1 \subseteq E_2 \subseteq E_3 \cdots$ is a countable family of nested sets and $E_i \in \mathcal{M}$, then μ satisfies *continuity from above*
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$
3. If $E_1 \supseteq E_2 \supseteq E_3 \cdots$ is a countable family of nested sets, $E_i \in \mathcal{M}$, $\mu(E_1) < \infty$, then μ satisfies *continuity from below*
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

3 Measure and Cardinality

Throughout this section and the next we discuss properties of ZFC.

As remarked in the introduction, injections carry measures to higher cardinalities while preserving their properties. We now make this notion precise. If $(X, \mathcal{P}(X), \mu)$ is a measure space on X and $X \preceq Y$, then we can define a measure λ on $\mathcal{P}(Y)$ as follows. Let $h : X \to Y$ be an injection and define $\lambda(E) = \mu(h^{-1}(E))$ for $E \subseteq Y$. We also recall, that in ZFC, many measure spaces must restrict to a subset of $\mathcal{P}(X)$ to be defined. Thus it is natural to ask: for which cardinals κ is there a measure μ such that $(\kappa, \mathcal{P}(\kappa), \mu)$ is a measure space on κ?

Without further restriction on what kinds of measures we are interested in, we see that $(\kappa, \mathcal{P}(\kappa), \gamma)$, where γ is counting measure, satisfies our criteria. This is uninteresting, several of the motivating examples of measures which cannot be defined on the full power set are *finite* measures, i.e. $\mu(X) < \infty$. Thus we ask: for which cardinals κ is there a finite measure defined on $\mathcal{P}(\kappa)$? Once again the answer is all cardinals. If κ is a cardinal, then $\emptyset \in \kappa$, so we define $\mu : \mathcal{P}(\kappa) \to [0, 1]$ by:

$$\mu(E) = \begin{cases} 1 & \text{if } \emptyset \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is then clear that μ is a finite measure. The question is made uninteresting because μ assigns singletons non-zero measure, we call such measures *trivial* in this paper. (Caveat: this notion is somewhere in between the standard definition of trivial (assigns all members of $\mathcal{M}$ measure zero) and of *atomic* (there is a set $A \in \mathcal{M}$ with $\mu(A) > 0$ such that for all $B \subset A, B \in \mathcal{M}$ implies $\mu(B) = 0$), since trivial measures are rarely discussed the author feels no great pain in recycling the terminology.)

Motivated by this, we refine our question again and ask: for which cardinals κ is there a non-trivial finite measure defined on $\mathcal{P}(\kappa)$? This is where the question becomes interesting. Immediately we see that no finite cardinals are acceptable. ω also fails, if $X \subseteq \omega$, has positive measure then by σ-additivity some $i \in X$ must have non-zero measure. Thus a cardinal satisfying our desiderata must be uncountable.

By our remarks above, the first cardinal with a non-trivial finite measure will be of great interest, as it induces such a measure on all larger cardinals.

4 Properties of Measurable Cardinals in ZFC

We begin with two remarkable properties of the smallest cardinal with a non-trivial finite measure. We note that these are theorems in ZFC, though we will later show we cannot prove the hypothesis "there is a smallest measurable cardinal". We follow the exposition of Drake [Dra74] in this section. The alternative in the first result comes from a case analysis; for a given measure either a standard 'splitting argument' can be used on this measure or it cannot.

Theorem 4.1. If κ is the smallest cardinal with a non-trivial finite measure on $\mathcal{P}(\kappa)$, then $\kappa \preceq 2^\omega$ or $\mathcal{P}(\kappa)$ has a non-trivial measure that takes only values in $\{0, 1\}$. We say such a measure is *two-valued*.

Proof. Let κ be the smallest cardinal such that $(\kappa, \mathcal{P}(\kappa), \mu)$ is a measure space and μ a non-trivial finite measure. For $A \subseteq \kappa$ such that $\mu(A) > 0$, we say that A *splits* if we can find disjoint sets A_1, A_2 where $A = A_1 \cup A_2$, and $0 < \mu(A_1) \leq \mu(A_2) < \mu(A)$. We treat the cases of splitting and non-splitting subsets separately.

If there is some $A \subseteq \kappa$ where $\mu(A) > 0$ and A does not split, then by σ-additivity, if $B \subseteq A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Then we define a measure on A by:

$$\nu(E) = \begin{cases} 1 & \text{if } \mu(E) = \mu(A) \\ 0 & \text{otherwise.} \end{cases}$$

Since μ is σ-additive and non-trivial ν shares these properties. By the minimality of κ we see $\kappa \preceq |A|$, so by the Cantor-Schröder-Bernstein theorem $|A| = \kappa$, and this induces a two-valued measure on κ.

Now we suppose that every subset of κ with nonzero measure splits. Since $\mu(\kappa) < \infty$ we can normalize μ and assume without loss of generality that $\mu(\kappa) = 1$. We will use this to construct a non-trivial finite measure on the space of countable zero-one sequences, ${}^\omega 2$, which has cardinality 2^ω, and so conclude $\kappa \preceq 2^\omega$.

First we show that if $A \subseteq \kappa$, $\mu(A) > 0$ we can split off large chunks of A.

Lemma 4.1. Let $(\kappa, \mathcal{P}(\kappa), \mu)$ be as above and suppose every subset of κ splits. Let $A \subseteq \kappa$. Then there is a partition $A = A_1 \cup A_2$ with $0 < \mu(A_1) \leq \mu(A_2)$ and $\mu(A_1) \geq \frac{1}{3}\mu(A)$.

Proof. Suppose not. Let

$$\delta = \frac{1}{\mu(A)} \sup\{\mu(A_1) | A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset, \mu(A_1) \leq \mu(A_2)\}.$$

Then for each n we can find an A_n such that

$$\mu(A)(\delta - \frac{1}{n}) < \mu(A_n) \leq \delta\mu(A).$$

Let $B = \cup_{n=1}^\infty A_n$. By continuity from below and the definition of δ, $\mu(B) = \delta\mu(A)$ and $A \setminus B$ cannot split (if it did we could contradict the definition of δ), a contradiction. $\Diamond$

Using the ability to split sets into large chunks we partition κ inductively using binary sequences. If $\emptyset$ is the binary sequence of length zero we let $\kappa_\emptyset = \kappa$. For a sequence s split κ_s into two parts, κ_{s0} and κ_{s1} such that $\frac{1}{3}\mu(\kappa_s) \leq \mu(\kappa_{s0}) \leq \mu(\kappa_{s1}) \leq \mu(\kappa_s)$. Then, for an infinite sequence s let s_n be the first n terms and define $\kappa_s = \cap_{n=1}^{\infty} \kappa_{s_n}$. By construction $\kappa_{s_n} \supseteq \kappa_{s_{n+1}}$ and $\mu(\kappa_{s_n}) \leq \left(\frac{2}{3}\right)^n$, so by continuity from above $\mu(\kappa_s) = 0$. Also, if $s \neq t$ are two binary sequences $\kappa_s \cap \kappa_t = \emptyset$, so $f(s) = \kappa_s$ is an injection from ${}^{\omega}2$ to μ measure zero subsets of κ. This map then naturally extends to $F : \mathcal{P}({}^{\omega}2) \to \mathcal{P}(\kappa)$, and $F({}^{\omega}2) = \kappa$. Thus $\nu(E) = \mu(F(E))$ is a measure on ${}^{\omega}2$ which is non-trivial and finite by construction. Hence $\kappa \preceq 2^{\omega}$. □

The smallest cardinal with non-atomic finite measure enjoys expanded additivity properties.

Theorem 4.2. If κ is the smallest cardinal with a non-trivial finite measure μ, then μ is κ-additive, i.e. if $\{E_\alpha\}_{\alpha \in A}$ is a family of pairwise disjoint subsets of κ and $A \prec \kappa$ then

$$\mu\left(\bigcup_{\alpha \in A} E_\alpha\right) = \sum_{\alpha \in A} \mu(E_\alpha).$$

Proof. First note that that the number of α such that $\mu(E_\alpha) > 0$ must be countable. Now suppose we have some family $\{E_\alpha\}_{\alpha \in A}$ with $A \prec \kappa$ where $\mu(\cup_{\alpha \in E_\alpha}) \neq \sum_{\alpha \in A} E_\alpha$. Since only countably many $\{E_\alpha\}_{\alpha \in B}$ $|B| = \omega$ have nonzero measure, we find

$$\begin{aligned} \infty > \mu\left(\bigcup_{\alpha \in B} E_\alpha\right) &= \sum_{\alpha \in B} \mu(E_\alpha) \\ &= \sum_{\alpha \in A} \mu(E_\alpha) \end{aligned}$$

and subtracting we conclude

$$\sum_{\alpha \in A} \mu(E_\alpha) - \sum \alpha \in B \mu(E_\alpha) = \sum_{\alpha \in A \setminus B} \mu(E_\alpha) = 0,$$

but $\mu(\bigcup_{\alpha \in A \setminus B} E_\alpha) = M > 0$. So we may assume without loss of generality that we have a family of sets $\{E_\alpha\}_{\alpha \in A}$ with $|A| \prec \kappa$, $\mu(E_\alpha) = 0$ but $\mu(\cup_{\alpha \in A} E_\alpha) = M > 0$. With this family we may define a measure ν on A by:

$$\nu(B) = \mu\left(\bigcup_{\alpha \in B} E_\alpha\right).$$

ν is finite since μ is, ν is non-trivial since for each $\alpha \in A$ $\nu(\{\alpha\}) = \mu(E_\alpha) = 0$, and ν is σ-additive. Indeed, if $\{F_n\}_{n=1}^{\infty}$ is a countable family of pairwise disjoint subsets of A then:

$$\begin{aligned} \sum_{n=1}^{\infty} \nu(F_n) &= \sum_{n=1}^{\infty} \mu\left(\bigcup_{\alpha \in F_n} E_\alpha\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} \bigcup_{\alpha \in F_n} E_\alpha\right) \text{ since } \mu \text{ is } \sigma\text{-additive} \\ &= \nu\left(\bigcup_{n=1}^{\infty} F_n\right), \end{aligned}$$

but $|A| \prec \kappa$, contradicting the minimality of κ. □

Since the smallest cardinal with a non-trivial finite measure κ induces a κ-additive measure on all larger cardinals, larger cardinals λ will only be of continued interest if they admit λ-additive measures. In light of this and Theorem 4.1 we define the following.

Definition 4.1. A cardinal $\kappa \succ \omega$ is *measurable* if κ admits a non-trivial κ-additive two-valued measure, and *real-valued measurable* if κ has a non-trivial finite κ-additive measure.

Remark 4.3: There is a non-trivial ω-additive (that is, finitely additive) two-valued measure on ω. Thus $\kappa \succ \omega$ is required, or we will have to deal with awkwardness either elsewhere in the definition or in proofs. As is standard in set theory we instead exclude ω from the definition.

We first note that (real-valued) measurable cardinals are *regular* in the sense that they are not the union of $\lambda \prec \kappa$ sets, each with cardinality strictly less than κ. (Note that regularity is the first condition of inaccessibility.)

Lemma 4.2. If κ is (real-valued) measurable, then κ is regular.

Proof. Let μ be a non-trivial κ-additive two valued measure on κ. If $X = \cup_{\alpha \in A} E_\alpha \subseteq \kappa$, with $|E_\alpha|, |A| \prec \kappa$, then since singletons have measure zero, by κ-additivity each E_α has measure zero, so X has measure zero by κ-additivity, hence $X \neq \kappa$. □

With this observation we can construct a rich measure on a measurable cardinal, further motivating our restriction to two-valued measures.

Theorem 4.4. If κ is measurable, then κ has a measure that takes on every value in $[0, 1]$.

Proof. We first find a countable family $\{A_n\}$ of disjoint subsets of κ such that $|A_n| = \kappa$. Since κ is a set of ordinals we define an equivalence relation $\asymp$ on κ by: $\alpha \asymp \beta$ if and only if $\beta = \alpha + n$ or $\alpha = \beta + n$ for some $n \in \omega$. Then we can partition κ into equivalence classes. Let E be an equivalence class, since E is a set of ordinals there is a least ordinal $\alpha \in E$. Then $E = \{\alpha + n\}_{n \in \omega}$, so each equivalence class is countable. Since κ is regular, there must be κ equivalence classes. Let $\{\alpha_\lambda\}_{\lambda \in \kappa}$ be the set of least ordinals in each equivalence class. Then set $A_n = \{\alpha_\lambda + n\}_{\lambda \in \kappa}$. By construction $|A_n| = \kappa$ and each A_n is disjoint.

Thus the non-trivial two-valued κ-additive measure μ on κ induces measures μ_n on each A_n. It is then easily seen

$$\nu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(A_n \cap E).$$

is the desired measure on κ. □

We are now ready to answer the question posed in Section 3 and show that in ZFC we cannot prove the existence of a measurable cardinal. (We will make further remarks regarding real-valued measures later.)

Theorem 4.5. If κ is measurable, κ is inaccessible.

Proof. We have already seen (Lemma 4.2) that κ is not the sum of less than κ cardinals all smaller than κ. It remains to show that if $\lambda \prec \kappa$, then $2^\lambda \prec \kappa$.

To do this we will show that if for some $\lambda \prec \kappa$ we have $2^\lambda \succeq \kappa$ then a κ-additive measure on κ must be trivial. Recall that ${}^\lambda 2$, the space of functions from λ to $\{0, 1\}$ has cardinality 2^λ, so if ν is a κ-additive two valued measure on κ it induces a measure μ on ${}^\lambda 2$. We define a function $f \in {}^\lambda 2$ by transfinite induction so that $\{f\}$ must have measure 1. To do this, for an ordinal $\beta \leq \lambda$ let $U(f, \beta) = \{g \in {}^\lambda 2 | f(\alpha) = g(\alpha), \alpha < \beta\}$. Note that for any ordinal β we can partition $U(f, \beta)$:

$$U(f, \beta) = \{\, g \in {}^\lambda 2 \,|\, f(\alpha) = g(\alpha), \alpha < \beta, g(\beta) = 0 \,\} \cup \{\, g \in {}^\lambda 2 \,|\, f(\alpha) = g(\alpha), \alpha < \beta, g(\beta) = 1 \,\}.$$

Note that $U(f, \emptyset) = {}^\lambda 2$. Since $\mu(U(f, \emptyset)) = 1$ for any f one of the two sets in the disjoint union above must have measure one. Call the two sets in the partition $U^0(f, \beta), U^1(f, \beta)$ respectively. Set $f(\emptyset) = i$ if and only if $\mu(U^i(f, \emptyset)) = 1$.

Suppose α is a successor ordinal and $\mu(U(f,\beta)) = 1$ for all $\beta \leq \alpha$. Then, since μ is two valued and $\mu(U(f,\alpha)) = 1$ we must have that one of $U^0(f,\alpha), U^1(f,\alpha)$ has measure 1, we set $f(\alpha) = i$ if and only if $\mu(U^i(f,\alpha)) = 1$. Then f is defined for all ordinals less than or equal to α and $\mu(U(f,\alpha+1)) = 1$.

If β is a limit ordinal and $\mu(U(f,\alpha)) = 1$ for all $\alpha < \beta$, define $f(\beta)$ as before, and note that:

$$U(f,\beta) = \cap_{\alpha<\beta} U(f,\alpha).$$

Using transfinite induction and κ-additivity we can extend continuity from above and below to the transfinite case, thus $\mu(U(f,\beta)) = 1$. So by transfinite induction f is defined as a function on λ taking values in $\{0,1\}$, $U(f,\lambda) = \{f\}$ and $\mu(U(f,\lambda)) = 1$, i.e. the measure ν must be trivial since h is an injection; a contradiction. Hence κ is inaccessible. □

It follows from our previous discussion of inaccessible cardinals (Section 2.4) that we cannot hope to prove, using ZFC, that there are measurable cardinals. We have not excluded the possibility of proving the existence of a real-valued measurable cardinal in ZFC. It can be shown that if κ is real-valued measurable then for any $\lambda \prec \kappa$ we have that $\aleph(\lambda) \prec \kappa$, this is known as being *weakly inaccessible* [Dra74]. If we assume the Generalized Continuum Hypothesis (GCH), that $2^{\aleph_\alpha} = \aleph_{\alpha+1} = \aleph(\aleph_\alpha)$, then the notions of weakly inaccessible and inaccessible coincide, and so ZFC+GCH cannot prove the existence of real-valued measurable cardinals.

5 Consistency of the existence of accessible measurable cardinals

Solovay has shown that there is a model of ZF in which every subset of reals is Lebesgue measurable; this naturally induces a real-valued measure on 2^ω, which is accessible [Sol70]. Thus it is consistent with ZF that there is an accessible real-valued measurable cardinal. The proofs in the previous section made heavy use of the Axiom of Choice (totally ordering cardinals by $\preceq$, partitioning equivalence classes), these two facts motivate the question: for which consequences A of the Axiom of Choice that are weaker than choice (i.e. $ZF + A \nvdash AC$) is ZF+A consistent with the existence of an accessible measurable cardinal?

The Axiom of Determinacy, proposed by Mycielski and Steinhaus [MS62], has some interesting consequences that help answer this question. We first state the axiom.

Definition 5.1. A *two-player ω-game with perfect information* is a game where two players alternate picking natural numbers forever, i.e. for each $n \in \omega$ there is an nth turn. This generates a sequence of natural numbers. The winner of the game is then decided by whether or not this sequence is in a set of sequences that win for player one, where the set is specified in advance and part of the game.

Definition 5.2. A two-player ω-game with perfect information A (A can be thought of as the set of winning sequences for player 1) is *determined* if player 1 has a winning strategy.

Axiom 10 (Determinacy). All two-player ω-games with perfect information are determined.

The Axiom of Determinacy is inconsistent with the Axiom of Choice, which makes its introduction somewhat controversial philosophically. However, it is a theorem in ZF+AD that $\aleph_1$ is measurable [Sol71]. ZF+AD also implies the following weak-choice form, known as countable choice [Bar89]:

Axiom 11 (Countable Choice).

$$\forall z(z \preceq \omega \wedge \forall x(x \in z \to \neg x = \emptyset \wedge \forall y(y \in z \to x \cap y = \emptyset \vee x = y)) \to \exists u \forall x \exists v(x \in z \to u \cap x = \{v\}))).$$

That is, for any countable family of sets we can perform the consequences of the axiom of choice. If we believe ZF+AD is consistent, then we must accept that ZF+CC is consistent with the existence of an accessible measurable cardinal. Checking Howard and Rubin's reference *Consequence of the Axiom of Choice* [HR91] quickly gives a list of choice-like principles that CC implies, giving further answers to our

question. Are there principles stronger than Countable Choice consistent with the existence of an accessible measurable cardinal? This question appears to be open [Zwa], and it is beyond the scope of this paper to begin answering it.

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References

[Bar89] Jon Barwise (ed.), *Handbook of mathematical logic*, Elsevier, 1989.

[Dev93] Keith Devlin, *The joy of sets: Fundamentals of contemporary set theory*, Undergraduate Texts in Mathematics, Springer, 1993.

[Dra74] Frank R. Drake, *Set theory: An introduction to large cardinals*, Studies in Logic and the Foundations of Mathematics, North-Holland, 1974.

[EFT94] H.-D. Ebbinghaus, J. Flum, and W. Thomas, *Mathematical logic*, Undergraduate Texts in Mathematics, Springer, 1994.

[Hal74] Paul R. Halmos, *Measure theory*, Graduate Texts in Mathematics, Springer, 1974.

[HR91] Paul Howard and Jean E. Rubin, *Consequences of the axiom of choice*, Mathematical surveys and monographs, American Mathematical Society, 1991.

[MS62] Jan Mycielski and H. Steinhaus, *A mathematical axiom contradicting the axiom of choice*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques **10** (1962).

[Sol70] Robert M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, The Annals of Mathematics, Second Series **92** (1970).

[Sol71] ———, *Real-valued measurable cardinals*, Axiomatic set theory (Proceedings of Symposia in Pure Mathematics, Volume XIII, Part I, University of California, Los Angeles, California, 1967), American Mathematical Society, Providence, R.I., 1971.

[Ula30] S. Ulam, *Zur Maßtheorie in der allgemeinen Mengenlehre.*, Fundamenta Mathematicae **16** (1930), 140–150 (German).

[Zwa] Cameron Zwarich, Personal Communication.

Atomic Realizations of Chemical Reaction Networks

Joseph Horan
University of Waterloo
jahoran@uwaterloo.ca

Abstract: When considering chemical reaction networks (CRNs), we often ignore the atomic compositions of the species involved (atom-free stoichiometry). Thus, a reasonable question is, "When can a CRN support an atomic realization?" That is, when can we assign each species an atomic structure which is distinct from that of the other species? After briefly covering definitions, we consider the main tool which we use, an algorithm from Schuster and Höfer [SH91], and draw out a proof that it obtains all extreme vectors. We then give necessary and sufficient conditions for atomic realizations, and discuss the implications, with reference to Famili and Palsson [FP03].

1 Introduction

In science, when explaining the world around us, we encounter systems involving different types of objects, species, elements, amongst other things. In chemistry, one such general system is what we call a *chemical reaction network*, or CRN; this is a set of individual chemical reactions. This system tells us about the participants in the reactions, or *species*, and how they react with each other over time, usually by way of differential equations. This is the field of *stoichiometry*. This is of interest primarily because it allows us to understand real-world processes. It turns out that while these systems tend to be extraordinarily complicated (in particular, highly non-linear), they are much more tractable than general systems, hence we wish to study them in a more in-depth fashion. First, let us briefly refresh ourselves on some of the terminology which we encounter; it is assumed that the reader has a grasp of basic linear algebra and differential equations, though some convex analysis will be explained.

A CRN is specified by four components: the species which participate in the set of reactions; the *stoichiometric coefficients*, usually given in matrix form as Γ, which describe the extent to which a species participates in a reaction; the *kinetics*, which describe the laws (usually differential equations) under which the species react; and the *rate constants*, which describe how quickly reactions take place. Ignoring the rates is reasonable for our purposes, as they are quite finicky and will not provide extra detail in the analysis to come. So, we can represent a chemical reaction network by

$$\sum_{i=1}^{n} \alpha_{ij} S_i \longrightarrow \sum_{i=1}^{n} \beta_{ij} S_i \ , \ j = 1, \ldots, r, \tag{1.1}$$

where each S_i is a chemical species, and the α_{ij}'s and β_{ij}'s are stoichiometric coefficients, all of which are non-negative. Notice that we merely denote a species by S_i, as opposed to an atomic formula. This is partially due to notation, but is also due to the fact that we tend to consider *atom-free* stoichiometries; that is, it is common to discuss the kinetics of the system without reference to the exact chemical formulae involved. Indeed, in some cases, this is the only option, as we may not know what formula a species actually has. Moreover, if two species have the same chemical formula, but different structure, we call them *isomers*. These make our job slightly more challenging.

Now, as per [HJ72], a *conservative* system is one for which there exists a positive vector in the nullspace of the transpose of the stoichiometric matrix. That is, $\exists z \in \text{null}(\Gamma^T) \cap \mathbb{R}^n_{>0}$. This definition encodes the

idea that the system maintains the total amount of species involved; in some sense, it is a first integral, to use DEs terminology. A consequence of this is that in conservative systems, the reactions must obey the *law of atomic balance*; that is, the total amount of a particular element must be conserved in the system, and on both sides of the reactions. So considering atom-free stoichiometries as opposed to ones with atomic structure leads to a certain loss of information here. We shall develop some mathematics to deal with atomic structure, and retrieve that information.

Treating the CRN as a dynamical system, one common and reasonable model is

$$\dot{x} = \Gamma R(x), \tag{1.2}$$

where $x \in \mathbb{R}^n_{\geq 0}$ is the species vector containing the concentrations of each species, Γ is the stoichiometric matrix defined by $\Gamma_{ij} = \gamma_{ij} = \beta_{ij} - \alpha_{ij}$, and $R(x)$ is the reaction rate vector, or kinetics of the system. For our purposes, we actually do not care what the kinetics are, but we do imagine them to be decently well-behaved (probably C^1).

Now, let $z \in \text{null}(\Gamma^T)$. Then we have:

$$z^T \dot{x} = z^T \Gamma R(x) = (\Gamma^T z)^T R(x) = 0^T R(x) = 0.$$

Integrating gives us:

$$z^T x = z^T x_0.$$

Then, if z is a positive vector and x_0 is non-negative, we see that all the species are bounded, and thus, by repeated application of Existence-Uniqueness, any solution of (1.2) is defined for all $t \geq 0$.

This is meaningful because in a way, each such vector z encodes a conservation relation for the species. The physical meaning of these vectors is not set in stone, but one interpretation is that they represent the number of a type of atom in each species. It then becomes important to know how to find these non-negative, non-zero (or semi-positive) left nullspace vectors.

2 Finding Semi-Positive Left Nullspace Vectors

We present an algorithm adapted by Schuster and Höfer in 1991 [SH91] from a Russian source [Che68] and a German source [FNB74], both of which are exceedingly difficult to find. We also verify that it does what it says it does.

We work in a finite-dimensional real vector space, to fit the situation. Recall that a *convex cone* is a set C which is closed under non-negative linear combinations, i.e. if $x, y \in C, \alpha, \beta \in \mathbb{R}_{\geq 0}$, then $\alpha x + \beta y \in C$. We hereby refer to convex cones as simply cones. One example of a cone is what is called an *orthant*, which is simply the generalization of a quadrant in the two-dimensional real plane, i.e. one of the 2^n segments of $\mathbb{R}^n$ where all the vectors in the orthant maintain their sign pattern as they move through the orthant. Clearly this property is preserved under non-negative linear combinations, so the orthant is a cone.

A cone then can be expressed in terms of a non-negative linear span of specific vectors in the cone; these vectors are called *extreme rays*, and they *generate* the cone, similar to the way a basis generates a subspace. We define a set of vectors to be *conically independent* in exactly the same way that a set of vectors is linearly independent, but with non-negative scalars. Considering cones is natural, because we desire that our solutions to remain within restrictions, such as the fact that all concentrations of species are non-negative, yet still possibly evolving over time in an unbounded fashion.

Finally, for a matrix A, denote $A_{|,j}$ and $A_{i,-}$ as the j^{th} column of A and the i^{th} row of A. For $n \in \mathbb{N}$, let $[n] = \{1, 2, ..., n\}$. For two vectors $x, y \in \mathbb{R}^n$, let $x \bullet y$ be the standard inner product of x and y, and for scalars a, b, let $a \cdot b$ be usual multiplication. This notation is used to be more clear.

Let $\Gamma = \Gamma^{(0)}$ be the stoichiometric matrix of a CRN. We define $T^{(0)}$ to be the *tableau* given by

$$T^{(0)} = \left[\Gamma^{(0)} \mid I_n \right].$$

The idea here is to take the non-negative orthant, which is a cone, and intersect it successively with each of the subspaces $\text{null}((\Gamma_{|,l})^T)$, each time obtaining a new cone. We construct vectors in each intersection from pairs of generating vectors from the previous cone, which will appear on the right-hand side of the tableau.

The inductive process goes as follows: Denote the i^{th} tableau as

$$T^{(i)} = \left[\Gamma^{(i)} \mid Z^{(i)} \right]. \tag{2.1}$$

For each tableau, let S_m be the set

$$S_m = \{a \in [n] \mid Z^{(i)}_{m,a} = 0\}. \tag{2.2}$$

If the j^{th} row of $T^{(i)}$ satisfies

$$\Gamma^{(i)}_{j,i+1} = 0, \tag{2.3}$$

then let

$$T^{(i+1)}_{l,-} = T^{(i)}_{j,-}.$$

If the j^{th} and k^{th} rows of $T^{(i)}$ satisfy

$$\Gamma^{(i)}_{j,i+1} \cdot \Gamma^{(i)}_{k,i+1} < 0, \tag{2.4}$$

$$S_j \cup S_k \nsubseteq S_m, \ \forall m \neq i, j, \tag{2.5}$$

then let a row in $T^{(i+1)}$ be given by

$$T^{(i+1)}_{l,-} = |\Gamma^{(i)}_{j,i+1}| \cdot T^{(i)}_{i,-} + |\Gamma^{(i)}_{i,i+1}| \cdot T^{(i)}_{j,-}.$$

Note that $T^{(i+1)}_{k,i+1} = 0$. This row creation is done as many times as is possible, and the resulting tableau is $T^{(i+1)}$. Condition (2.5) yields conically independent vectors on the right-hand side of the tableau, in the rows of $Z^{(i)}$. In many cases, there will be fewer rows in the new tableau than in the old tableau. If the new tableau is empty because there are no possible combinations, then the algorithm terminates and there are no conservation relations for the system. Finally, the algorithm has completed when $\Gamma^{(r)}$ is reached, which will be an p-by-r matrix of zeroes. Then, the generators/extreme rays of the cone are the p rows of $Z^{(r)}$.

We now spend some time to actually prove that this is the case.

Claim 2.1. The resulting vectors (the p rows of $Z^{(r)}$) are contained in $\text{null}(\Gamma^T)$. Furthermore, they generate the cone $\text{null}(\Gamma^T) \cap \mathbb{R}^n_{\geq 0}$.

Proof. We proceed by induction, as alluded to above.

Base case: The rows of I_n are $e_j, j \in [n]$, and we note that any possible non-negative combination remains in the non-negative orthant. If $\Gamma^{(0)}_{j,1} = 0$, $e_j \in \text{null}(\Gamma^T_{|,1})$, and moreover, since e_j only has one non-zero coordinate and will independent from any other vectors obtained, it is clearly a generator for the cone $\text{null}(\Gamma^T_{|,1}) \cap \mathbb{R}^n_{\geq 0}$. Then, if for two indices j, k conditions (2.4), (2.5) are satisfied, the resulting vector $Z^{(1)}_{l,-}$ from above is indeed in $\text{null}(\Gamma^T_{|,1})$, since we have

$$\begin{aligned} Z^{(1)}_{l,-} \bullet \Gamma_{|,1} &= (|\Gamma^{(i)}_{j,1}| \cdot \Gamma^{(i)}_{k,-} + |\Gamma^{(i)}_{k,1}| \cdot \Gamma^{(i)}_{j,-}) \bullet \Gamma_{|,1} \\ &= |\Gamma^{(i)}_{j,1}| \cdot \Gamma^{(i)}_{k,1} + |\Gamma^{(i)}_{k,1}| \cdot \Gamma^{(i)}_{j,1} \\ &= 0, \end{aligned}$$

seeing as we chose the row combination explicitly for this. As well, the vector will be a generator for the cone, since the only vectors with the same sign pattern also in the cone are multiples. Else, by taking differences, we could obtain a vector with only one non-zero component in the nullspace, which in the current case is impossible.

In general: We have the i^{th} tableau $T^{(i)}$, non-empty, and that the vectors $Z^{(i)}_{j,-}$ generate the cone $\cap_{s=1}^{i}\text{null}((\Gamma_{|,s})^T) \cap \mathbb{R}^n_{\geq 0}$. We construct $T^{(i+1)}_{k,-}$ by either one of the methods in the algorithm, and note that, for example,

$$\begin{aligned} 0 = T^{(i+1)}_{k,i+1} &= \Gamma^{(i+1)}_{k,i+1} \\ &= |\Gamma^{(i)}_{j,i+1}| \cdot \Gamma^{(i)}_{k,i+1} + |\Gamma^{(i)}_{k,1}| \cdot \Gamma^{(i)}_{j,i+1} \\ &= |\Gamma^{(i)}_{j,i+1}| \cdot (|\Gamma^{(i-1)}_{j_1,i+1}| \cdot \Gamma^{(i-1)}_{k_1,i+1} + |\Gamma^{(i-1)}_{k_1,1}| \cdot \Gamma^{(i-1)}_{j_1,i+1}) + \dots \\ &\vdots \\ &= \sum_{s=1}^{n} Z^{(i+1)}_{k,s}\Gamma_{s,i+1} = Z^{(i+1)}_{k,-} \bullet \Gamma_{|,i+1}, \end{aligned}$$

since the entries in $Z^{(i+1)}$ are exactly the factors involved in the row scaling and combinations. In the other case, it can trace back to a situation like above, or the vector is an e_j. Hence in all cases, $Z^{(i+1)}_{k,-} \in \text{null}((\Gamma_{|,i+1})^T)$, as desired.

Now, suppose those vectors do not generate the whole cone. Let v be one of the other generators. Since v must be in the previous cone, we can write it

$$v = \sum_k v_k Z^{(i)}_{k,-},\ v_k \geq 0\ \forall\ k.$$

Then, note that we can find a non-negative combination of $Z^{(i+1)}_{j,-}$'s such that it can be written as a non-negative combination of the same extreme rays as v, with potentially different coefficients. As before, taking scaled differences can now take us out of the nullspace, which is a contradiction. Thus v cannot be a generator for the cone. Therefore, by induction, our claim is proven. □

It should be noted that any resulting conservation vectors will have integer entries, since $\Gamma \in M_{n\times r}(\mathbb{Z})$, and we are performing non-negative combinations with integer scalings. This is important in the following section. As well, taking a combination of all the vectors, say just adding all of them together, yields a strictly positive vector in the cone, which gives us the situation as described at the end of section 1. This happens because for each coordinate index, at least one generating vector will have a non-zero entry in that coordinate, because we formed vectors by taking non-negative combinations of the standard basis vectors.

3 Atomic Realizations

With the algorithm under our belts now, we consider the atomic implications of having conservation relations for the system. As stated in the introduction, usually we deal with atom-free stoichiometries. [ET89] is an occurrence in the literature where atomic stoichiometry is considered, though it's only for a couple of sections in the book; they then move on to atom-free stoichiometry. However, we can use the convex basis of conservation vectors to identify atomic balances between the species, and thus identify the structure of said species.

Specifically, let $Z^{(r)}$ be the matrix consisting of the conservation vectors as rows, satisfying the condition that all columns are distinct. Then, to each row assign a distinct element, and to each column assign a species. Reading the column then yields the possible atomic structures of that species. This is an *atomic realization* of the system. The stated condition means that no two species have the same atomic structure. Hence, we are excluding isomers from this discussion, since we want to look at atomic structure and composition only, and isomers fog the picture.

We note that we can pick and choose of the conservation relations to get a realization, and in particular, there exist realizations with a minimum number of distinct elements. This is important, because we can then

hypothesize, for a given reaction system with species of unknown compositions, what those compositions might be, and decide on a reasonable atomic realization for the model. One thing to note is that it could be that we would have a matrix with distinct columns, but we could take a subset of the rows and end up with non-distinct columns. Thus systems can have minimal requirements as to the complexity of the atomic compositions of the species.

Keeping in mind that the cone has non-empty interior, we could add those two vectors together to get a positive vector, and presumably the one-element atomic relation would be the simplest realization possible, if the vector had distinct entries. This is correct, but ultimately less than satisfactory. Here, this is better interpreted as conservation of atoms/etc. Good judgement should be used when performing analysis, as always.

This discussion differs from the discussion found in Famili and Palsson [FP03], for instance, because they already have structure to their species, and they can tailor their interpretations to the structure. Here, we have minimal structure knowledge, and thus we cannot guess. Given information, of course, we can interpret the relations differently.

We provide an example to illustrate. Consider the system

$$\begin{aligned} A + 2B &\longrightarrow C + D \\ B + C &\longrightarrow A + E. \end{aligned}$$

Performing the algorithm yields, in order:

$$T^{(0)} = \left[\begin{array}{cc|ccccc} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$T^{(1)} = \left[\begin{array}{cc|ccccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$T^{(2)} = \left[\begin{array}{cc|ccccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 \end{array}\right].$$

We observe that the four conservation relations do have distinct columns, so the system has an atomic realization involving four different elements, namely:

$$A = WX, \quad B = XYZ, \quad C = WZ_2, \tag{3.1}$$

$$D = X_3Y_2, \quad E = YZ_3. \tag{3.2}$$

We also note that any three of the four relations taken together have distinct columns as well, but that the majority of ways to take two of the four together either yield a column of zeroes (that is, a species has no atomic structure at all), or have non-distinct columns. However, taking rows two and four give a possible two-element realization:

$$A = X, \quad B = XY, \quad C = Y_2, \tag{3.3}$$

$$D = X_3, \quad E = Y_3. \tag{3.4}$$

This is a minimal atomic realization, and provides the simplest structure hypothesis. Finally, note that convex integer combinations could possibly be an atomic realization, but taking the generating vectors of the cone capture the idea that atoms are indivisible, as opposed to functional groups.

4 Conclusion

We reviewed the idea of conservation relations in CRN theory, and proved that a given algorithm for finding them works effectively. We then introduced the idea of atomic realizations, in the hopes that this provides yet another tool in the analysis of CRNs, by restricting the possible atomic structures available for each species.

5 Acknowledgements

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References

[Che68] S. Chernikov, *Linear inequalities (in russian)*, Nauka, Moscow, 1968.

[ET89] P. Érdi and J. Tóth, *Mathematical models of chemical reactions*, Manchester University Press, Manchester, 1989.

[FNB74] J. Guddat F. Nožička, H. Hollatz and B. Bank, *Theorie der linearen parametrischen optimierung*, Akademie-Verlag, Berlin, 1974.

[FP03] I. Famili and B. Palsson, *The convex basis of the left null space of the stoichiometric matrix leads to the definition of metabolically meaningful pools*, Biophysical Journal (2003).

[HJ72] F. Horn and R. Jackson, *General mass action kinetics*, Arch. Rational Mech. Anal. (1972).

[SH91] S. Schuster and T. Höfer, *Determining all extreme semi-positive conservation relations in chemical reaction systems: A test criterion for conservativity*, J. Chem. Soc. Faraday Trans (1991).

Asymptotic eigenvalue distribution of random lifts

Jean-Philippe Fortin, Samantha Rudinsky
McGill University
jp.fortin@mail.mcgill.ca
samantha.rudinsky@mail.mcgill.ca

Abstract: A *random n-lift* $L_n(G)$ of a base graph G is obtained by replacing each vertex v_i of G by a set V_i of n vertices, and generating a random matching between V_i and V_j for each edge $(v_i, v_j) \in G$. We show that the spectral density of a random lift $L_n(G)$ approaches that of a tree as we increase n by showing that the expected number of short cycles of length k in $L_n(G)$ (denoted $Z_k(G)$) tends to a constant λ_k. Moreover, we show that $Z_k(G)$ is Poisson distributed with parameter λ_k. We also give experimental results of the level spacing distributions and compare them to the Gaussian Orthogonal Ensemble of random matrix theory.

1 Introduction

A regular graph is a graph where each vertex has exactly the same degree, i.e. the same number of adjacent vertices. For a d-regular graph G, let $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$ be the eigenvalues of the adjacency matrix (defined in section 2). It can be easily shown that $\eta_1 = d$ and $|\eta_i| \leq d$ for the remaining eigenvalues. Let $\rho(G) = \max(|\eta_2|, |\eta_n|)$ be the second-largest eigenvalue of the graph G. Define the edge expansion constant $h(G)$ to be

$$h(G) = \min_{0 \leq |A| \leq n/2} \frac{|\partial A|}{|A|}$$

where A is any nonempty subset having at most $n/2$ vertices, and ∂A is the set of edges with exactly one endpoint in A. The edge expansion comes from the theory of expander graphs, where one wishes to construct an efficient network with a good connection property. Networks can be seen as vertices sharing data via edges connecting them. The edge expansion constant measures how well connected is the network by restricting the number of wires used in the network, but at the same time ensuring that any subset A is well connected to its complement $\bar{A}$. In the case G is a d-regular graph, the edge expansion constant is related to $\rho(G)$ by the following bounds derived by Dodziuk, Alon-Milman and Alon [Alo86, AM85, Dod84]:

$$\frac{d - \rho(G)}{2} \leq h(G) \leq \sqrt{2d(d - \rho(G))}$$

The quantity $d - \rho(G)$ is defined as the spectral gap of the graph G. If follows that the edge expansion of G is particularly significant when ρ is small. Moreover, a theorem derived by Alon and Boppana [Alo86, Nil91] states that $\rho(G) \geq 2\sqrt{(d-1)} - o_n(1)$. In the optimal case where $\rho(G) \leq 2\sqrt{d-1}$, the graph G is said to be Ramanujan. An open question is to prove the existence of infinite families of d-regular Ramanujan graphs for all $d \geq 3$ [ABG10], which would provide an infinite family of optimal expanders. The idea of random lifts was introduced by Friedman [Fri03] in order to obtain new Ramanujan graphs from smaller ones (namely base graphs). Addario-Berry and Griffiths showed [ABG10] that with extremely high probability, all eigenvalues of the random lift that are not eigenvalues of the base graph have order $O(\sqrt{d})$. This implies that if the base graph is Ramanujan, then the random lift is with high probability nearly Ramanujan.

In the original paper introducing random lifts [ALMR01], a variety of properties of random lifts are discussed: connectivity, expansion, independent sets, colouring and perfect matchings. In the present

article, we first study the asymptotic eigenvalue distribution of random n-lifts as $n \to \infty$. We show that it follows the asymptotic law given in a paper by McKay [McK81]:

$$f(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} & \text{for } |x| \leq 2\sqrt{d-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The main part of the proof is to show that the number of short cycles of a fixed length k (denoted Z_k) tends to a constant, denoted λ_k, which depends only on the base graph G. By following the method of moments described by Janson, Luczak and Rucinsky [JLK00], we show precisely that Z_k is Poisson distributed according to the parameter λ_k.

The second part of our article is about the level spacing distribution of random lifts. First, define the sequence of unfolded eigenvalues to be $x_i = \{F(\eta_i)\}$, where F is the cumulative function associated with the asymptotic density (1.1). Then the quantities $s_i = x_{i+1} - x_i$ are called the spacings of the graph G. By following the work of Jakobson, Miller, Rivin and Rudnick [JMRR99], we give experimental results about the level spacing distribution of random lifts, and we show that there is a good fit between the random lift spacings and the Gaussian Orthogonal Ensemble (GOE) spacings. The GOE comes from the theory of random matrices; it is claimed that eigenvalues of large random symmetric matrices model the fluctuations of energy levels of chaotic dynamical systems [BGS84].

2 Short cycles in random lifts

2.1 Graphs and random lifts

We denote the set of vertices and the set of edges of graph G by $V(G)$ and $E(G)$ respectively. By definition, two vertices v_1 and v_2 are adjacent (or neighbors) if they are connected by an edge $e \in E(G)$ and the degree of a vertex v, denoted $\deg(v)$, is the number of edges adjacent to v. For the rest of the paper, we consider only simple graphs G such that all vertices have a fixed degree d. We call such graphs *d-regular simple graphs*. Note that simple graphs are graphs that contain no loops or parallel edges. An important tool in the study of graphs is the adjacency matrix $A(G)$ which is an $n \times n$ matrix, with $n = |V(G)|$, where a_{ij} is the number of edges from v_i to v_j. In the case of simple d-regular graphs, all diagonal entries of A are zero and the remaining entries are either 0 or 1. Moreover, the sum of the entries of each row and each column is equal to d.

We define a k-cycle to be a connected 2-regular subgraph whose edges and vertices are only traversed once, therefore containing k vertices and k edges. A closed walk of length k is defined as a sequence of adjacent vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ so that the first and last vertices are the same, i.e $v_1 = v_{k+1}$. A closed non-backtracking walk is defined as a closed walk such that for any vertex $v_i \in \{v_3, \ldots, v_{k+1}\}$, we have $v_i \neq v_{i=2}$. Conversely, a closed backtracking walk is a closed walk such that $v_i = v_{i-2}$ for at least one $i \in \{2, \ldots k+1\}$.

A *random n-lift* $L_n(G)$ of a graph G is obtained by replacing each vertex v_i of G by a set V_i of n vertices (called the fibre of v_i), and placing a random matching between V_i and V_j for each edge $(v_i, v_j) \in G$. We call G the *base graph* of the lift.

2.2 Short cycles in random lifts

Let $L_n(G)$ be a random n-lift of the d-regular graph G. Define Z_k as the number of cycles of length k in $L_n(G)$ for $k \geq 3$. Define c_k as the number of closed non-backtracking walks of length k in G. We first show that the expected number of k-cycles for a fixed k approaches a constant which depends only on the base graph as $n \to \infty$:

Lemma 2.1. $\mathbb{E}(Z_k) \to \frac{c_k}{2k}$ as $n \to \infty$.

Proof. Let p_k be the probability that a subset of k edges occurs in the lift $L_n(G)$. We will show that $p_k \sim \frac{1}{n^k}$ as $n \to \infty$, i.e. $\lim_{n\to\infty} \frac{p_k}{1/n^k} = 1$. Let Γ_k be the subgraph formed by the k edges and let $\pi(\Gamma_k)$ be the projection of Γ_k on the base graph G. For an edge $e \in \pi(\Gamma_k)$, let $m(e)$ be the number of edges in Γ_k projected on e. We call $m(e)$ the *multiplicity of the edge* e. Let s_i be the number of edges $e \in \pi(\Gamma_k)$ such that $m(e) \geq i$ and let r be the greatest multiplicity of an edge in $\pi(\Gamma_k)$. Then we have

$$s_r = k - \sum_{i=1}^{r-1} s_i$$

For an edge $e \in \pi(\Gamma_k)$ with multiplicity m, we have m corresponding edges in the subgraph Γ_k within the same fibre. This set of edges occurs with probability $\frac{1}{n}\frac{1}{n-1}\cdots\frac{1}{n-(m-1)}$. It follows that

$$\begin{aligned}
p_k &= \prod_{i=1}^{r}\left(\frac{1}{n+1-i}\right)^{s_i} \\
&= \left(\prod_{i=1}^{r-1}\left(\frac{1}{n+1-i}\right)^{s_i}\right)\left(\frac{1}{n+1-r}\right)^{k-\sum_{i=1}^{r-1} s_i} \\
&= \left(\prod_{i=1}^{r-1}\left(\frac{n+1-r}{n+1-i}\right)^{s_i}\right)\left(\frac{1}{n+1-r}\right)^{k} \\
&\sim \frac{1}{n^k} \text{ as } n \to \infty
\end{aligned}$$

Let w_k be a cycle of length k in $L_n(G)$. The projection $\pi(w_k) \in G$ must be a closed non-backtracking walk in the base graph G. Fix a closed non-backtracking walk $w'_k \in G$. Let $a(w'_k)$ be the number of possible cycles $w_k \in L_G(n)$ where $\pi(w_k) = w'_k$. We will show that $a(w'_k) \sim \frac{n^k}{2k}$ as $n \to \infty$.

Let t_i be the number of vertices which appear at least i times in w'_k, for $i = 1, 2, \ldots, l$ where l is the maximal occurrence of a vertex in w'_k. We have

$$t_l = k - \sum_{i=1}^{l-1} t_i$$

To count the number of possible cycles $w_k \in L_G(n)$, we count how many ways we can choose the vertices in w_k such that their projections are the vertices of w'_k. For a vertex u_j which appears for the first time in the walk, we have n choices for choosing a vertex of w_k in the fibre V_{u_j}. When a vertex u_j appears for its i-time, we have $(n-i+1)$ choices in V_{u_j} since $(i-1)$ vertices have already been chosen in this fibre. Since there are $2k$ possible ways to start the process, we have

$$\begin{aligned}
2ka(w'_k) &= \prod_{i=1}^{l}(n+1-i)^{t_i} \\
&= \left(\prod_{i=1}^{l-1}(n+1-i)^{t_i}\right)(n+1-l)^{k-\sum_{i=1}^{l-1} t_i} \\
&= \left(\prod_{i=1}^{l-1}\left(\frac{n+1-i}{n+1-l}\right)^{t_i}\right)(n+1-l)^{k} \\
&\sim n^k \text{ as } n \to \infty
\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}(Z_k) &= \sum_{w'_k} a(w'_k)p_k \\ &\sim \sum_{w'_k} \frac{n^k}{2k}\frac{1}{n^k} \text{ as } n \to \infty \\ &= \frac{c_k}{2k}\end{aligned}$$

which proves the lemma.

Following the same idea for a general subgraph H of $L_n(G)$, one has the following lemma:

Lemma 2.2. Let H be a subgraph of $L_n(G)$ with v vertices and e edges. Then the expected number of such subgraphs in $L_n(G)$ is $O(n^{v-e})$. In the case $e > v$, we have $\mathbb{E}(H) = O\left(\frac{1}{n}\right)$.

Proof. The idea is similar to that of the proof of Lemma 1. We have already shown that the probability that a subset of e edges occurs in the random lift is $p^e \sim n^{-e}$. Moreover, as n increases, the number of possible vertices that can be chosen among a fibre, for a fixed vertex of the projection $\pi(H)$ on G, is of order n, i.e. $a(\pi(H)) = O(n^v)$ for any subgraph H containing v vertices. One concludes by noticing that taking the sum over the finite number of subgraphs H' such that $\pi(H) = H'$ does not affect the result for the asymptotic result, i.e.

$$\sum_{H' \in G : \pi(H') = H} \frac{1}{n^e} O(n^v) = O(n^{v-e})$$

Janson, Luczak and Rucinsky showed in [JLK00] that for random regular graphs, the number of cycles of length k is distributed according to a Poisson distribution with parameter $\theta_k = \frac{1}{2k}(d-1)^k$. They used the method of moments in the case of Poisson distributions and used specifically the following theorem:

Theorem 2.1 (Theorem 6.10 of [JLK00]). Let $(X_n^{(1)}, \ldots, X_n^{(m)})$ be vectors of non-negative and bounded random variables, where $m \geq 1$ is fixed. If $\lambda_1, \ldots, \lambda_m \geq 0$ are such that, as $n \to \infty$,

$$\mathbb{E}((X_n^{(1)})_{k_1} \ldots (X_n^{(m)})_{k_m}) \to \lambda_1^{k_1} \ldots \lambda_m^{k_m}$$

for every $k_1, \ldots, k_m \geq 0$, where $\mathbb{E}(X)_i$ denotes the i-th factorial moment of X, then $(X_n^{(1)}, \ldots, X_n^{(m)}) \to^d (Z_1, \ldots, Z_m)$, where $Z_i \in Po(\lambda_i)$ are independent Poisson variables.

We will show a similar result for random n-lifts:

Theorem 2.2. Let $\lambda_k := \frac{c_k}{2k}$, where c_k is the number of closed non-backtracking walks in the base graph G, and let $Z_{k\infty} \in \text{Poisson}(\lambda_k)$ be independent Poisson distributed random variables, $k = 1, 2, 3, \ldots$ Then the random variables $Z_k(L_n(G))$ converge in distribution to $Z_{k\infty}$, i.e. $Z_k(L_n(G)) \to^d Z_{k\infty}$ as $n \to \infty$, jointly for all k.

Proof. We need to compute the factorial moments $\mathbb{E}(Z_k)_i$ for all $i \geq 2$. We begin with $\mathbb{E}(Z_k)_2$. By definition $\mathbb{E}(Z_k)_2 = \mathbb{E}(Z_k(Z_k - 1))$, i.e. $\mathbb{E}(Z_k)_2$ is the expected number of pairs of two distinct cycles of length k. We write $\mathbb{E}(Z_k)_2 = Y' + Y''$ where Y' is the number of pairs of vertex disjoint cycles and Y'' is the number of pairs of two cycles with at least one common vertex. We can decompose Y'' further according to the number of common vertices and to the number of total edges in the pair. Then $Y'' = \sum_{j=1}^{J} Y''_j$ where J depends only on k. Since the Y''_j count the number of some subgraphs which have more edges than vertices, we have $\mathbb{E}(Y''_j) = O\left(\frac{1}{n}\right)$ for all j by Lemma 2. It follows that $\mathbb{E}(Y'') = O\left(\frac{1}{n}\right)$ since J does not depend on n. It remains to show that $\mathbb{E}(Y') \to \lambda_k^2$. We proceed in a similar way as we did for $\mathbb{E}(Z_k)$ in Lemma 1:

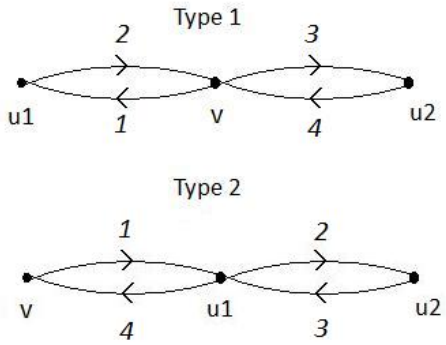

Figure 2.1: Backtracking closed walks of length 4 starting at the vertex v. Numbers represent the steps of the walk.

We have $p_{2k} \sim \frac{1}{n^{2k}}$. Let $w_{kk} = w_k^{(1)} \sqcup w_k^{(2)}$ be a pair of two disjoint k-cycles in $L_n(G)$. The projected walks $\pi(w_k^{(1)})$ and $\pi(w_k^{(2)})$ are two closed non-backtracking k-walks in G which may intersect or not. Moreover, it is possible that $\pi(w_k^{(1)}) = \pi(w_k^{(2)})$. Let $w'_{kk} = \pi(w_k^{(1)}) \sqcup \pi(w_k^{(2)})$. For any $w'_{kk} \in G$, let $d(w'_{kk})$ be the number of pairs of two disjoint k-cycles $w_{kk} \in L_n(G)$ such that $w'_{kk} = \pi(w_{kk})$. As $n \to \infty$, we have $d(w'_{kk}) \sim \left(\frac{n^k}{2k}\right)^2$. Summing over all possible pairs $w'_{kk} \in G$, we get $\sum_{w'_{kk}} d(w'_{kk}) \sim \left(\frac{c_k n^k}{2k}\right)^2$. If follows that $\mathbb{E}(Y') \sim (\lambda_k)^2$ and $\mathbb{E}(Z_k)_2 \sim (\lambda_k)^2$. The same argument applies on any factorial moment $\mathbb{E}(Z_k)_i$ and for any combination $\mathbb{E}((Z_k)_{k_1} \dots (Z_k)_{k_m})$. By Theorem 1, the proof is complete.

To illustrate this result, we give as an example the simple case where $G = K_{d+1}$, the complete d-regular graph on $(d+1)$ vertices.

To count the number of non-backtracking walks of length k, we use the following idea. First, we choose the first vertex of the walk, which gives $d+1$ possibilities. For the second vertex, we have d possibilities. For the third vertex, since backtracking is not allowed, we are left with only $(d-1)$ possibilities. In general, for the i-th vertex, with $2 \leq i \leq k-2$, we have $(d-1)$ possibilities as well. For $i = k-1$, we cannot choose the initial vertex of the walk neither, since this would imply backtracking at the end of the walk; we therefore have $(d-3)$ choices for the $(k-1)$-th vertex. It follows that

$$\lambda_k = \frac{c_k}{2k} = \frac{(d+1)d(d-1)^{k-3}(d-3)}{2k}$$

Let A be the adjacency matrix of the graph $L_n(G)$. It is easy to see that $\mathrm{tr}(A^k)$ gives the total number of closed walks of length k in the graph $L_n(G)$. We will analyze the case $k = 4$. To compute $Z_4(L_n(G))$ from the adjacency matrix, we need a relation between $Z_4(L_n(G))$ and $\mathrm{tr}(A^4)$. To do so, we need to subtract from $\mathrm{tr}(A^4)$ the number of closed walks which are not cycles. Fix a vertex $v \in L_n(G)$. There are two types of backtracking closed walks of length 4, presented in Figure 1.

For the first type, there are d^2 such walks since there are d ways of choosing v_1 and d ways of choosing v_2. For the second type, there are $d(d-1)$ such walks. Since there are $(d+1)n$ vertices in $L_n(G)$, we have the following relation:

$$Z_4 = \frac{tr(A^4) - nd(d+1)(2d-1)}{8}$$

We divided the right-hand side by 8 since every walk of length 4 is being counted exactly 8 times: 4 ways to fix a vertex and 2 ways to traverse the cycle. We computed Z_4 for 500 random lifts of the complete graph K_9 with $n = 200$ and compared the empirical distribution with the asymptotic Poisson distribution of Z_4. By using the formula above, we get λ_4 to be 378 $(9 \times 7 \times 6)$. By looking at the density of the Poisson

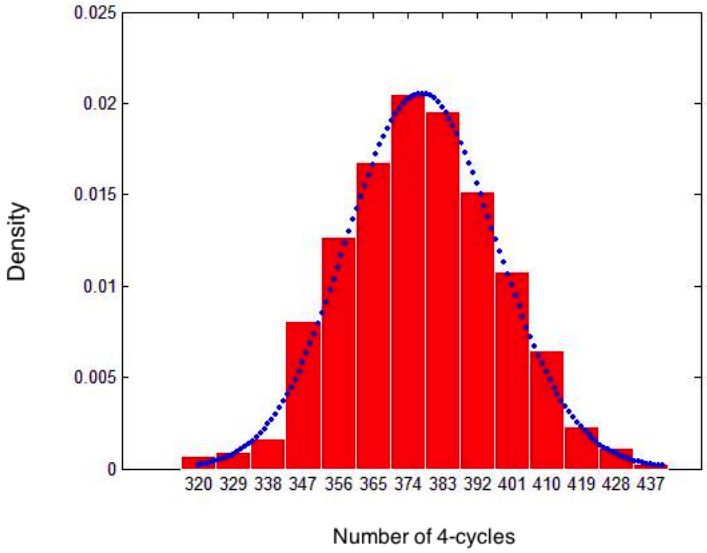

Figure 2.2: Number of 4-cycles for 500 n-lifts of the complete graph K_9 with $n = 200$ vs. the theoretical Poisson distribution of Z_4.

distribution with $\lambda = 378$ against the empirical distribution (see Figure 2), we observe that the expected behavior of Z_4 for large n is a very good fit for n as small as 200.

3 Asymptotic eigenvalue distribution of random lifts

We will study here the eigenvalues of $L_n(G)$ where G is a d-regular graph and when $n \to \infty$. We notice first that the eigenvalues of the base graph G are all inherited by $L_n(G)$. To see this, let η be an eigenvalue of $A(G)$ with eigenvector x. Define the vector y in the following way: $y_v = x_i$ if $v \in V_i$ and where V_i is the fibre of the vertex $u_i \in G$. Then y is an eigenvector or $L_n(G)$ with eigenvalue η. In McKay [McK81], the following theorem is proved:

Theorem 3.1. Let $X_1, X_2, \ldots$ be a sequence of regular graphs, each of degree $v \geq 2$, which satisfies the conditions

- $n(X_i) \to \infty$ as $i \to \infty$ where $n(X_i)$ is the number of vertices of the graph X_i
- for each $k \geq 3$, $\frac{Z_k(X_i)}{n(X_i)} \to 0$ as $i \to \infty$

Let $f(X_i, x)$ be the density distribution of the eigenvalues of $A(X_i)$. Then for each $x, f(X_i, x) \to f(x)$ as $i \to \infty$, where $f(x)$ is the function defined as follows:

$$f(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} & \text{for } |x| \leq 2\sqrt{d-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We refer to $f(x)$ as *McKay's law*. Since we proved previously that $\mathbb{E}(Z_k) \to \lambda_k(G)$ as $n \to \infty$, we have

$$\mathbb{E}\left(\frac{Z_k}{|L_n(G)|}\right) \sim \frac{\lambda_k(G)}{nd} \to 0 \text{ as } n \to \infty,$$

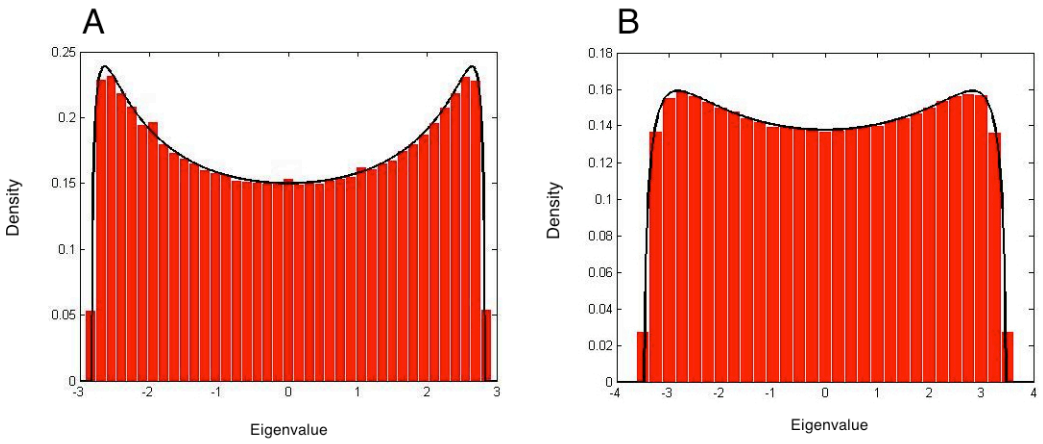

Figure 4.1: A) Eigenvalue distribution of 200-lifts of the complete graph K_5 vs McKay's law B) Eigenvalue distribution of 200-lifts of a random 3-regular graph on 6 vertices vs McKay's law

which shows that the second condition of the theorem is satisfied. We conclude that the asymptotic eigenvalue distribution of random lifts follows McKay's law.

4 Experimental results

We computed the empirical eigenvalue distributions of two ensembles of graphs obtained by lifting two different base graphs: the first is a ensemble of 200-lifts of the complete graph K_5 (see Figure 3A) and the second is a ensemble of 200-lifts of a random generated 3-regular graph with 6 vertices (see Figure 3B). In both figures, the empirical distributions are compared to McKay's law. The reader has to notice that we did not include the old eigenvalues of the lifts (the eigenvalues inherited from the base graph) in the distributions.

In [JMRR99], it is conjectured that the level spacing distribution of random regular graphs is similar to that of the Gaussian Orthogonal Ensemble (GOE), which is a statistical model in Random Matrix Theory. The empirical level spacing distribution is obtained in the following way: We first unfold the spectrum by setting

$$x_j = F(\eta_j)$$

where $F(x)$ is the cumulative distribution function associated to McKay's law. Then the sequence of numbers $\{x_j\}$ has unity as mean spacing. We consider the spacings $s_n = x_{n+1} - x_n$. The distribution function of the s_n is called the *level spacing distribution*. For the GOE, an approximation derived by Wigner is known for the level spacing distribution (called *Wigner surmise*):

$$P_W(s) = \frac{\pi}{2} s e^{\frac{-\pi s^2}{4}}$$

We computed the level spacing distribution of our previous graphs and we plotted the results in comparison to the Wigner surmise (Figure 4). The results show a good fit and lead us to think that the eigenvalues of random lifts have GOE spacings.

5 Random lift generation

For the experiments, we used three different base graphs: complete graphs K_{d+1}, complete bipartite graphs $K_{d,d}$ and random d-regular graphs. For generating random d-regular graphs, we used the configuration

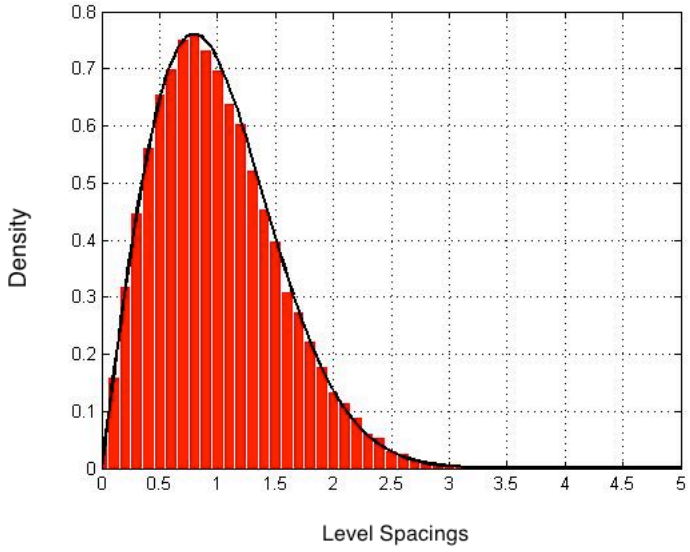

Figure 4.2: Level spacing distribution of 100-lifts of a random 3-regular graph on 6 vertices *vs* GOE Wigner surmise

model (Bollobas) described in [JLK00]. Briefly, an $2d$-configuration of a graph G is a partition of the cartesian product $W = \{1, \ldots, |G|\} \times \{1, \ldots 2d\}$ into dn pairs, where $|G| = n$ is the number of vertices of the graph G. The natural projection of the configuration W onto G creates a d-regular graph G'.

Now, to construct a random n-lift H from a base graph G, construct first the $nv \times nv$ matrix A which represents the adjacency matrix of H, where $v = |V(G)|$. The v first columns (or rows) represent the vertices of the first copy of G. The next v columns represent the vertices of the second copy of G, and so on. For each edge $(i, j) \in G$, we construct an array X of size n. For the k-th entry of X, we generate a random number Rand(k) between 0 and m where m is much larger than n. We sort the array X in such a way that we keep track of the original indices. Denote the new index of Rand(k) by $\pi(k)$. The associations k to $\pi(k)$ create a perfect matching between the fibres V_i and V_j. For each pair $(k, \pi(k))$, we set

$$A_{(i+kn, j+\pi(k)n)} = 1 \text{ and } A_{(j+\pi(k)n, i+kn)} = 1$$

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References

[ABG10] Louigi Addario-Berry and Simon Griffiths, *The spectrum of random lifts*, arXiv preprint arXiv:1012.4097 (2010).

[ALMR01] Alon Amit, Nathan Linial, Jiří Matoušek, and Eyal Rozenman, *Random lifts of graphs*, Proceedings of the twelfth annual ACM-SIAM symposium on Discrete algorithms, Society for Industrial and Applied Mathematics, 2001, pp. 883–894.

[Alo86] Noga Alon, *Eigenvalues and expanders*, Combinatorica **6** (1986), no. 2, 83–96.

[AM85] Noga Alon and Vitali D Milman, *λ¡ sub¿ 1¡/sub¿, isoperimetric inequalities for graphs, and superconcentrators*, Journal of Combinatorial Theory, Series B **38** (1985), no. 1, 73–88.

[BGS84] Oriol Bohigas, Marie-Joya Giannoni, and Charles Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Physical Review Letters **52** (1984), no. 1, 1–4.

[Dod84] Jozef Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Transactions of the American Mathematical Society **284** (1984), no. 2, 787–794.

[Fri03] Joel Friedman, *Relative expanders or weakly relatively ramanujan graphs*, Duke Mathematical Journal **118** (2003), no. 1, 19–35.

[JLK00] Svante Janson, Tomasz Luczak, and VF Kolchin, *Random graphs*, Cambridge Univ Press, 2000.

[JMRR99] Dmitry Jakobson, Stephen D Miller, Igor Rivin, and Zeév Rudnick, *Eigenvalue spacings for regular graphs*, Emerging Applications of Number Theory, Springer, 1999, pp. 317–327.

[McK81] Brendan D McKay, *The expected eigenvalue distribution of a large regular graph*, Linear Algebra and its Applications **40** (1981), 203–216.

[Nil91] Alon Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics **91** (1991), no. 2, 207–210.

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