A brief introduction to measurable cardinals

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ABSTRACT: We introduce the notion of a measurable cardinal, motivated by examples from measure theory. We then develop some initial inaccessibility results for such cardinals and summarize Solovay's results regarding the consistency of set theory with the hypothesis that there is an accessible measurable cardinal. These results lead us to a question that appears to be open (it is certainly open ended) regarding the consistency of weaker forms of choice and the existence of measurable cardinals. This paper aims to be self contained and accessible to an advanced undergraduate; however, its motivation rests in measure theory so previous exposure to the subject will be helpful.

1 INTRODUCTION

Measure theory is the study of assigning size to subsets of a given set (the assignments are known as measures). Measures are required to interact sensibly with set operations, though a large portion of measure theory is concerned with measures compatible with other structures on a space. Note that injections commute with set operations. Therefore, when a measure is defined on every subset of a set we can obtain a measure on any larger cardinality. In light of this, in the search for measures with interesting properties, we can restrict ourself to the smallest cardinality admitting a measure with a given property.

Counting measure (the size of a set is its cardinality) and Dirac measures (a subset is assigned size infinity if it contains a given element and size zero otherwise) assign a size to every subset, but they are uninteresting in most applications of measure theory. Interesting examples of measures (Lebesgue measure on $\mathbb{R}^n$, the natural probability measure on $\{0,1\}^{\mathbb{N}}$, Radon measures on uncountable locally compact Hausdorff spaces) must exclude a class of problematic subsets to be defined. This leads us to ponder: what makes these measures 'interesting'? And is there a cardinality high enough to admit an interesting measure defined on the power set of a set of that cardinality? Since the smallest cardinal with an interesting measure will provide interesting measures on larger cardinalities we will give it particular attention.

Cardinals that admit such measures were first considered by Ulam [Ula30]. The central result regarding such cardinals affirms the belief that measure theory must make do with non-measurable sets and work around them. To be specific, it is known that if the commonly accepted Zermelo-Frankel set theory with the axiom of choice (ZFC) is consistent then we cannot prove, using ZFC, that *any* cardinality admits an interesting measure. Both this result and classical results regarding the existence of non-measurable sets rely on the Axiom of Choice; we are lead to suspect it is the source of the trouble.

Solovay explored this line of reasoning, and showed that there is a model of ZF (the Zermelo-Frankel axioms without choice) set theory where the axiom of choice does not hold and every subset of the real numbers is Lebesgue measurable [Sol70]. In the interest of making this paper accessible to readers not well versed in modern set theory we omit a detailed discussion of Solovay's results. Further results of Solovay show that ZF with a weaker variants of choice are consistent with interesting measures defined on the power set of familiar cardinals [Sol71].

This paper is organized as follows. Section 2 introduces the necessary background for this paper, both set theory and measure theory. The exposition is necessarily terse and proofs are omitted, but it serves to make the paper self contained. Readers already familiar with these topics will not be harmed by reading lightly here. Section 3 explores formally the interesting properties of a measure desired for our investigation. Section 4 develops the central result of the paper, that in ZFC we cannot hope to prove that any cardinality admits an interesting measure. Section 5 discusses briefly the consistency of the existence of cardinalities possessing nice measures in ZF set theory with weaker versions of choice.

2 Background: Zermelo-Frankel set theory and basic measure theory

2.1 Formal systems, models, and incompleteness

In taking after Hilbert, if we regard mathematics as a formal game played symbolically with an alphabet we can then analyze this game mathematically. (A philosophically inclined reader may object to this particular portrayal. A detailed discussion of the philosophy at play here is very far afield indeed, so we acknowledge the objection, ignore it, and move on.) To this end we introduce the language of first order logic. The alphabet is $(\ )\ \wedge\ \vee\ \rightarrow \leftrightarrow\ \neg\ =\ \forall\, \exists\, v_0\, v_1\, \ldots$. The list of variables is understood to be countable. When discussing a particular bit of mathematics we enrich this base alphabet with constant symbols (conventionally $c_0, c_1, \ldots$), relation symbols (denoted $R_0, R_1, \ldots$), and function symbols (denoted $f_0, f_1, \ldots$). Function and relation symbols are specified along with a fixed arity for each symbol. We restrict our attention to strings in this language that mirror those we use in mathematics regularly, the syntax mirrors standard notation (though brackets are mandatory), and these are known as *well-formed formulas*. Proofs are modeled by a set of rules for combining basic proof steps, known as a sequent calculus, these rules mirror logical deduction like one would expect (for example, from $(\phi \rightarrow \psi) \wedge \phi$ we may conclude ψ). If all variables in a well-formed formula (string with valid syntax) are preceded with a quantifier ($\forall$ or $\exists$) they are said to be *bound* and such a formula is known as a *statement*. Any variable not preceded by a quantifier is said to be *free*.

Definition 2.1. A *language* is the set of all formulas that can be formed using a fixed collection of constant, relation, and function symbols.

Definition 2.2. A *formal system* is some collection of statements Σ in a language. The statements in Σ are usually referred to as *axioms*.

Definition 2.3. A statement ϕ is *derivable* in a formal system Σ if from Σ we can create ϕ with finitely many applications of first order logic rules. Such a ϕ is a *theorem* of Σ, and we write $\Sigma \vdash \phi$.

Definition 2.4. A formal system Σ is *consistent* if there is no statement ϕ such that $\Sigma \vdash \phi \wedge \neg\phi$.

Definition 2.5. A *model* of a formal system M is a set along with assignments of the constant symbols to elements of M, the relation symbols to subsets of direct products of M, and the function symbols to functions from direct products of M to M such that every statement in Σ is true in M (the quantifiers are understood to range over M).

We note that if a system has a nonempty model then it must be consistent.

An important result in the theory of formal systems is Gödel's Second Incompleteness theorem. Using a numeric encoding of statements of a formal system (the binary of the ASCII values of the characters in the string will work, provided the encoding is unique), if a system Σ can encode and prove the axioms of standard arithmetic it can be self referential in the following fashion. If n^ϕ is the numeric encoding of ϕ we can define, in arithmetic, the statement $\mathrm{Der}_\Sigma(n^\phi)$ that is true if $\Sigma \vdash \phi$. Then, if in Σ we can prove the axioms of arithmetic, the statement $\neg\mathrm{Der}_\Sigma(n^{0=1})$, denoted Consis_Σ is a statement in the language of Σ. Gödel's theorem can then be stated:

Theorem 2.1 (Second Incompleteness). If a formal system Σ (with computable axioms) can encode and prove the axioms of standard arithmetic and Σ is consistent then it is not the case that $\Sigma \vdash \mathrm{Consis}_\Sigma$.

Our discussion here is necessarily lacking in formality and detail. The theory of formal systems is a field of study in its own right and tends to be very verbose to state in full detail. The interested reader is referred to any standard introduction to the subject, such as the book by Ebbinghaus, Flum, and Thomas [EFT94], for a detailed development of the theory.

2.2 The axioms of Zermelo-Frankel set theory

We now introduce the formal system taken by most mathematicians as the formal foundation of mathematics, the system of Zermelo-Frankel set theory. We present the axioms both as a convenient reference and a brief introduction. We include an informal discussion of each axiom. The language used has no constant symbols or function symbols and a single relation symbol $\in$. For clarity we will use x, y, z, u, v, w for variables, it is understood that we can re-write this only in terms of v_i, the actual variable symbols.

Axiom 1 (Extensionality).

$$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y).$$

This axiom requires sets to be defined by their elements.

Axiom 2 (Empty Set).

$$\exists x \forall y(\neg y \in x).$$

This axiom specifies the existence of *an* empty set. We denote a set with this property $\emptyset$, but caution that $\emptyset$ is not a constant symbol in our language, and it may not be unique in all models.

Axiom 3 (Pair).

$$\forall x, y \exists z \forall w(w \in z \leftrightarrow w = x \lor w = y).$$

Typical notation for z is $\{x, y\}$, and $\{x\}$ for $\{x, x\}$. (Note that the axioms, as written do not give us a method for creating "the set containing x", and even if such an axiom were included it would be redundant, as we would find $\{x\} = \{x, x\}$ by Extensionality.)

Axiom 4 (Union).

$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists t(z \in t \land t \in x)).$$

This axiom states that y is the union of all elements of x. Using Axiom 3 we can construct $z = x \cup y$.

Axiom 5 (Infinity).

$$\exists x(\emptyset \in x \land \forall y(y \in x \to y \cup \{y\} \in x)).$$

We may construct the natural numbers by taking $0 = \emptyset$ and using the successor function $x \mapsto \{x, \{x\}\}$. This axiom then guarantees us a set of all natural numbers. This axiom is required to guarantee an infinite set; there are models of the other axioms where every set is finite. In light of this a natural question, that our discussion will touch on, is whether or not infinities beyond the reach of these axioms exist.

The next axiom is not a single axiom, but an axiom *schema*, a computable process for generating countably many axioms.

Axiom 6 (Replacement). If ϕ is a formula with at least two free variables:

$$\forall t_1, t_2, \dots t_k (\forall x \exists ! y \phi(x, y, t_1, \dots, t_k) \to \forall u \exists v B(u, v)),$$

where $B(u, v)$ is the formula:

$$\forall r (r \in v \leftrightarrow \exists s (s \in u \wedge \phi(s, r))).$$

The notation $\exists!$ is for "there exists a unique" and can be formed in first order logic. The hypothesis of this axiom is that ϕ encodes a partial function, its conclusion is that the range of ϕ over sets is also a set. This is indeed a countable schema, there are countably many strings over our alphabet, and only some subset of them are formulas with two free variables.

Axiom 7 (Power Set).

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall u (u \in z \to u \in x)).$$

This axiom guarantees a power set for every set. While it may appear that this is subsumed by replacement the formula not quantified does not code a function, consider $x = \emptyset$. Indeed, Power Set implies an uncountable set and there is a model of the other axioms in which every set is countable.

Axiom 8 (Regularity).

$$\forall x \exists y (x = \emptyset \vee (y \in x \wedge \forall z (z \in x \to \neg z \in y))).$$

This is a technical axiom that is not used directly in higher mathematics. It specifies that as a partial order, $\in$ is always well-founded; i.e. there is an element that is minimal with respect to $\in$. (This does not make $\in$ a well-order on every set.) This axiom serves to exclude $x \in x$ and other paradox inducing sets. The reader already familiar with set theory may protest that we have left out the axiom schema of separation. No such omission has been made, separation is a consequence of empty set and replacement [Dev93].

These eight axioms make up what we will refer to as system ZF. There is another axiom taken by most modern mathematicians, the Axiom of Choice.

Axiom 9 (Choice).

$$\forall x (x \in z \to \neg x = \emptyset \wedge \forall y (y \in z \to x \cap y = \emptyset \vee x = y)) \to \exists u \forall x \exists v (x \in z \to u \cap x = \{v\}).$$

We note that $x \cap y$ can be defined using Replacement; for a fixed x the set $x \cap y$ is the image of y under $\phi(y, z) = \forall u (u \in z \leftrightarrow u \in x \wedge u \in y)$. Informally, Choice reads "if x is a family of sets then there is a set u made of one of each element of the family x". Note that Choice allows us to make potentially uncountably many choices, even if we have no concrete property ϕ that we can appeal to Replacement with. We denote system ZF with the addition of Choice by ZFC.

Recalling our discussion of incompleteness, we remark that ZF can prove the axioms of standard arithmetic, and so cannot prove $\text{Consis}_{\text{ZF}}$; similarly ZFC cannot prove $\text{Consis}_{\text{ZFC}}$, assuming these systems are consistent. Since our definition of a model is in terms of sets (treated intuitively, not as parts of a formal system), we can use this coding scheme to represent informal sentences about models of ZFC as formal statements about *sets in* ZFC. We can also code and prove as a theorem, inside ZFC, that the existence of a nonempty model implies consistency. It then follows from the incompleteness theorem that ZFC cannot be consistent *and* prove that a model of ZFC exists.

2.3 Ordinal numbers

In set theoretic investigations the ordinal numbers provide a useful generalization of the order and induction properties of the natural numbers. In the interests of brevity we will simply state a few results about ordinals that will be useful later, the interested reader can check any standard set theory introduction [Dev93].

Definition 2.6. A *well-ordering* on a set A is a relation $<$ such that for all $x, y \in A$ exactly one of $x < y, y < x$, or $x = y$ is true, and if $x, y, z \in A$ and $x < z, z < y$ then $x < y$, and A has a least element with respect to this order.

Definition 2.7. A set is *transitive* if $\forall z \in A$ we have $y \in z \to y \in A$. That is, $z \subseteq A$.

Definition 2.8. An *ordinal* is a transitive set α well-ordered by $\in$.

We attempt to capture the natural hierarchy of well-ordering structures with this definition. One quickly sees that $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ are all ordinals (we denote this sequence $0, 1, 2, \ldots$ and take it to be the definition of the natural numbers) and that in general if α is an ordinal then $\alpha \cup \{\alpha\}$ is an ordinal.

Theorem 2.2. An ordinal is the set of all ordinals that precede it.

Remark 2.3: This lets us order the ordinals by $\in$, and it well-orders them.

Theorem 2.4. If α is an ordinal then $\beta = \alpha \cup \{\alpha\}$ is the least ordinal greater than α, we write $\beta = \alpha + 1$.

Theorem 2.5. If S is a set of ordinals then there is a least ordinal in S.

Theorem 2.6. If S is a set of ordinals then there is a least ordinal α such that $\beta \in S \to \beta < \alpha$. We write $\alpha = \sup S$.

Definition 2.9. α is a *successor* ordinal if there is an ordinal β such that $\alpha = \beta + 1$. α is a *limit ordinal*] otherwise.

Theorem 2.7. There exists a limit ordinal. The least limit ordinal is the set of the natural numbers, denoted ω.

The following theorem generalizes induction and recursion to higher infinities.

Theorem 2.8 (Transfinite Induction). Let $P(\alpha)$ be a property defined on the ordinals. Suppose that $P(\emptyset)$ is true and $P(\alpha)$ holding for all $\alpha < \beta$ implies $P(\beta)$. Then P is true for all ordinals.

Note that we often treat the cases of limit and successor ordinals separately when conducting proof by transfinite induction.

2.4 Cardinal numbers

Cardinal numbers are used to discuss the relative size of sets. The notions of injective, surjective, and bijective function can be defined in ZF, so the following discussion makes sense in ZF. If there is a bijection between two sets we say they are similar and we denote this $x \sim y$. We also say that $x \preceq y$ if $x \sim z$ for some $z \subseteq y$. We note that in ZF we can show that $x \sim y$ is an equivalence relation and that if $z \subseteq x \subseteq y$, $z \preceq x \preceq y$.

Theorem 2.9 (Cantor-Schröder-Bernstein). ZF $\vdash (x \preceq y \wedge y \preceq x) \leftrightarrow x \sim y$, or in English, if there are injections $x \to y$ and $y \to x$ then x is similar to y.

Definition 2.10. An ordinal α is an *initial ordinal* if for all $\beta < \alpha$, $\neg(\beta \sim \alpha)$, that is, it is not similar to any earlier ordinal.

Definition 2.11. $\aleph(x) = \{\alpha | \alpha \preceq x\}$. The aleph function collects all ordinals similar to a subset of its argument.

Theorem 2.10. For any ordinal α, $\aleph(\alpha)$ is an ordinal, and for an initial ordinal $\aleph(\alpha)$ is the next initial ordinal.

Since initial ordinals create new levels in the order $\preceq$ we define:

Definition 2.12. A *cardinal* is an initial ordinal.

Using the $\aleph$ function and transfinite induction we can define a hierarchy of increasingly large cardinals

$$\begin{aligned} \aleph_0 &= \omega \\ \aleph_{\alpha+1} &= \aleph(\aleph_\alpha) \\ \aleph_\beta &= \bigcup_{\alpha<\beta} \aleph_\alpha \text{ for limit ordinals } \beta. \end{aligned}$$

In addition to the aleph function we can use a theorem of Cantor to get larger cardinals.

Theorem 2.11. In ZF $\neg(\mathcal{P}(x) \preceq x)$.

In words, the power set of a cardinal is a set of larger cardinality. By analogy with finite sets, for a cardinal κ we write 2^κ for the cardinal similar to $\mathcal{P}(\kappa)$.

For an arbitrary set let $|x| = \cap\{\alpha \,|\, \alpha \sim x,\ \alpha \text{ is an ordinal}\}$. If x is not similar to any ordinal then $|x| = \emptyset$, thought this cannot happen in ZFC—in ZFC every set is well-order-able and therefore similar to some ordinal. It is easy to check that $|x|$ is an initial ordinal, that for initial ordinals $|\alpha| = \alpha$, and that if $|x| \neq \emptyset$ and $x \sim y$ then $|x| = |y|$. We call $|x|$ the *cardinality* of x.

We can define a notion of cardinal addition; for an index set I we define $\sum_{i\in I} \kappa_i = \lambda$ by

$$\sum_{i\in I} \kappa_i = \left| \coprod_{i\in I} \kappa_i \right|,$$

where II stands for disjoint union, though we cannot show that this union is nonempty without the Axiom of Choice. We also remark that $\preceq$ is not necessarily a total order on sets in the absence of Choice.

With the notion of cardinal defined we now introduce a class of peculiarly large cardinals that will play a role later in the paper.

Definition 2.13. A cardinal κ is *inaccessible* if:

1. If $\lambda \prec \kappa$, κ cannot be written $\Sigma_{i\in\lambda}\theta_i$ for $\theta_i \prec \kappa$; that is, if κ is a cardinal sum, then either one summand is already of cardinality κ or there are at least κ summands

2. If $\lambda \prec \kappa$ then $2^\lambda \prec \kappa$.

ω is inaccessible, and in fact is a model of the axioms of ZFC other than Infinity. Since larger inaccessibles contain ω, similar reasoning shows that these higher cardinals are models of ZF and ZFC. (An example of the aforementioned reasoning, the second condition implies that for any proper subset $A \subset \kappa$ something behaving like the power set $\mathcal{P}(A)$ can also be found as a subset of κ, so the Power Set axiom is satisfied.)

2.5 Measure theory

We now briefly recall the relevant definitions and results form measure theory for the unfamiliar reader. While this paper aims to be accessible with no further measure theory, the examples of measure spaces given in a standard reference, such as Halmos's [Hal74], are occasionally referred to and will help the reader put the discussion in context.

Let X be a set.

Definition 2.14. We say a family of sets $\mathcal{M} \subseteq \mathcal{P}(X)$ is a *σ-algebra* (over X, if not clear from context) if

1. $\emptyset \in \mathcal{M}$.
2. $E \in \mathcal{M}$ implies $X \setminus E \in \mathcal{M}$.
3. If $E_1, E_2, \ldots \in \mathcal{M}$, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

   For any set X, $\mathcal{P}(X)$ is a σ-algebra.

Definition 2.15. A function $\mu : \mathcal{M} \to [0, \infty]$ defined on a σ-algebra $\mathcal{M}$ is a *measure* if

1. $\mu(\emptyset) = 0$.
2. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ when $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. This property is known as *σ-additivity.*

Definition 2.16. The triple $(X, \mathcal{M}, \mu)$ for a set X, σ-algebra $\mathcal{M}$, and measure μ on $\mathcal{M}$ is a *measure space on X.*

Example 2.1. Let $a \in X$ be an arbitrary element. Then $(X, \mathcal{P}(X), \delta_a)$ where

$$\delta_a(E) = \begin{cases} \infty & \text{if } a \in E \\ 0 & \text{otherwise.} \end{cases}$$

is a measure space on X. δ_a is known as a *Dirac measure* on X.

Example 2.2. It is easily seen that for $(X, \mathcal{P}(X), \gamma)$ where

$$\gamma(E) = \begin{cases} n & \text{if } |E| = n \\ \infty & \text{if } \omega \preceq E \end{cases}$$

is a measure space on X. γ is known as the *counting measure* on X.

Example 2.3. Let $\mathcal{M}$ be the collection of countable and co-countable (those sets $E \subset X$ such that $X \setminus E$ is countable) subsets of X. Then $\mathcal{M}$ is a σ-algebra and $(X, \mathcal{M}, \mu)$ where $\mu(E) = 1$ if E is co-countable and zero otherwise is a measure space.

Example 2.4. Let $(\Omega, \mathcal{B}, P)$ be a sample space. Probability theorists call σ-algebras *σ-fields*, and it is readily seen that the probability of some event (subset of Ω in the σ-field) is a measure.

The following theorem summarizes the basic properties of measures that will be used in this paper.

Theorem 2.12. If $(X, \mathcal{M}, \mu)$ is a measure space, then:

1. If $E, F \in \mathcal{M}$, $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. If $E_1 \subseteq E_2 \subseteq E_3 \cdots$ is a countable family of nested sets and $E_i \in \mathcal{M}$, then μ satisfies *continuity from above*
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$
3. If $E_1 \supseteq E_2 \supseteq E_3 \cdots$ is a countable family of nested sets, $E_i \in \mathcal{M}$, $\mu(E_1) < \infty$, then μ satisfies *continuity from below*
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

3 Measure and Cardinality

Throughout this section and the next we discuss properties of ZFC.

As remarked in the introduction, injections carry measures to higher cardinalities while preserving their properties. We now make this notion precise. If $(X, \mathcal{P}(X), \mu)$ is a measure space on X and $X \preceq Y$, then we can define a measure λ on $\mathcal{P}(Y)$ as follows. Let $h : X \to Y$ be an injection and define $\lambda(E) = \mu(h^{-1}(E))$ for $E \subseteq Y$. We also recall, that in ZFC, many measure spaces must restrict to a subset of $\mathcal{P}(X)$ to be defined. Thus it is natural to ask: for which cardinals κ is there a measure μ such that $(\kappa, \mathcal{P}(\kappa), \mu)$ is a measure space on κ?

Without further restriction on what kinds of measures we are interested in, we see that $(\kappa, \mathcal{P}(\kappa), \gamma)$, where γ is counting measure, satisfies our criteria. This is uninteresting, several of the motivating examples of measures which cannot be defined on the full power set are *finite* measures, i.e. $\mu(X) < \infty$. Thus we ask: for which cardinals κ is there a finite measure defined on $\mathcal{P}(\kappa)$? Once again the answer is all cardinals. If κ is a cardinal, then $\emptyset \in \kappa$, so we define $\mu : \mathcal{P}(\kappa) \to [0, 1]$ by:

$$\mu(E) = \begin{cases} 1 & \text{if } \emptyset \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is then clear that μ is a finite measure. The question is made uninteresting because μ assigns singletons non-zero measure, we call such measures *trivial* in this paper. (Caveat: this notion is somewhere in between the standard definition of trivial (assigns all members of $\mathcal{M}$ measure zero) and of *atomic* (there is a set $A \in \mathcal{M}$ with $\mu(A) > 0$ such that for all $B \subset A, B \in \mathcal{M}$ implies $\mu(B) = 0$), since trivial measures are rarely discussed the author feels no great pain in recycling the terminology.)

Motivated by this, we refine our question again and ask: for which cardinals κ is there a non-trivial finite measure defined on $\mathcal{P}(\kappa)$? This is where the question becomes interesting. Immediately we see that no finite cardinals are acceptable. ω also fails, if $X \subseteq \omega$, has positive measure then by σ-additivity some $i \in X$ must have non-zero measure. Thus a cardinal satisfying our desiderata must be uncountable.

By our remarks above, the first cardinal with a non-trivial finite measure will be of great interest, as it induces such a measure on all larger cardinals.

4 Properties of Measurable Cardinals in ZFC

We begin with two remarkable properties of the smallest cardinal with a non-trivial finite measure. We note that these are theorems in ZFC, though we will later show we cannot prove the hypothesis "there is a smallest measurable cardinal". We follow the exposition of Drake [Dra74] in this section. The alternative in the first result comes from a case analysis; for a given measure either a standard 'splitting argument' can be used on this measure or it cannot.

Theorem 4.1. If κ is the smallest cardinal with a non-trivial finite measure on $\mathcal{P}(\kappa)$, then $\kappa \preceq 2^\omega$ or $\mathcal{P}(\kappa)$ has a non-trivial measure that takes only values in $\{0, 1\}$. We say such a measure is *two-valued*.

Proof. Let κ be the smallest cardinal such that $(\kappa, \mathcal{P}(\kappa), \mu)$ is a measure space and μ a non-trivial finite measure. For $A \subseteq \kappa$ such that $\mu(A) > 0$, we say that A *splits* if we can find disjoint sets A_1, A_2 where $A = A_1 \cup A_2$, and $0 < \mu(A_1) \leq \mu(A_2) < \mu(A)$. We treat the cases of splitting and non-splitting subsets separately.

If there is some $A \subseteq \kappa$ where $\mu(A) > 0$ and A does not split, then by σ-additivity, if $B \subseteq A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Then we define a measure on A by:

$$\nu(E) = \begin{cases} 1 & \text{if } \mu(E) = \mu(A) \\ 0 & \text{otherwise.} \end{cases}$$

Since μ is σ-additive and non-trivial ν shares these properties. By the minimality of κ we see $\kappa \preceq |A|$, so by the Cantor-Schröder-Bernstein theorem $|A| = \kappa$, and this induces a two-valued measure on κ.

Now we suppose that every subset of κ with nonzero measure splits. Since $\mu(\kappa) < \infty$ we can normalize μ and assume without loss of generality that $\mu(\kappa) = 1$. We will use this to construct a non-trivial finite measure on the space of countable zero-one sequences, ${}^{\omega}2$, which has cardinality 2^{ω}, and so conclude $\kappa \preceq 2^{\omega}$.

First we show that if $A \subseteq \kappa$, $\mu(A) > 0$ we can split off large chunks of A.

Lemma 4.1. Let $(\kappa, \mathcal{P}(\kappa), \mu)$ be as above and suppose every subset of κ splits. Let $A \subseteq \kappa$. Then there is a partition $A = A_1 \cup A_2$ with $0 < \mu(A_1) \leq \mu(A_2)$ and $\mu(A_1) \geq \frac{1}{3}\mu(A)$.

Proof. Suppose not. Let

$$\delta = \frac{1}{\mu(A)} \sup\{\mu(A_1) | A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset, \mu(A_1) \leq \mu(A_2)\}.$$

Then for each n we can find an A_n such that

$$\mu(A)(\delta - \frac{1}{n}) < \mu(A_n) \leq \delta\mu(A).$$

Let $B = \cup_{n=1}^{\infty} A_n$. By continuity from below and the definition of δ, $\mu(B) = \delta\mu(A)$ and $A \setminus B$ cannot split (if it did we could contradict the definition of δ), a contradiction. $\Diamond$

Using the ability to split sets into large chunks we partition κ inductively using binary sequences. If $\emptyset$ is the binary sequence of length zero we let $\kappa_{\emptyset} = \kappa$. For a sequence s split κ_s into two parts, κ_{s0} and κ_{s1} such that $\frac{1}{3}\mu(\kappa_s) \leq \mu(\kappa_{s0}) \leq \mu(\kappa_{s1}) \leq \mu(\kappa_s)$. Then, for an infinite sequence s let s_n be the first n terms and define $\kappa_s = \cap_{n=1}^{\infty} \kappa_{s_n}$. By construction $\kappa_{s_n} \supseteq \kappa_{s_{n+1}}$ and $\mu(\kappa_{s_n}) \leq \left(\frac{2}{3}\right)^n$, so by continuity from above $\mu(\kappa_s) = 0$. Also, if $s \neq t$ are two binary sequences $\kappa_s \cap \kappa_t = \emptyset$, so $f(s) = \kappa_s$ is an injection from ${}^{\omega}2$ to μ measure zero subsets of κ. This map then naturally extends to $F : \mathcal{P}({}^{\omega}2) \to \mathcal{P}(\kappa)$, and $F({}^{\omega}2) = \kappa$. Thus $\nu(E) = \mu(F(E))$ is a measure on ${}^{\omega}2$ which is non-trivial and finite by construction. Hence $\kappa \preceq 2^{\omega}$. $\square$

The smallest cardinal with non-atomic finite measure enjoys expanded additivity properties.

Theorem 4.2. If κ is the smallest cardinal with a non-trivial finite measure μ, then μ is κ-additive, i.e. if $\{E_\alpha\}_{\alpha \in A}$ is a family of pairwise disjoint subsets of κ and $A \prec \kappa$ then

$$\mu\left(\bigcup_{\alpha \in A} E_\alpha\right) = \sum_{\alpha \in A} \mu(E_\alpha).$$

Proof. First note that that the number of α such that $\mu(E_\alpha) > 0$ must be countable. Now suppose we have some family $\{E_\alpha\}_{\alpha \in A}$ with $A \prec \kappa$ where $\mu(\cup_{\alpha \in E_\alpha}) \neq \sum_{\alpha \in A} E_\alpha$. Since only countably many $\{E_\alpha\}_{\alpha \in B}$ $|B| = \omega$ have nonzero measure, we find

$$\begin{aligned} \infty > \mu\left(\bigcup_{\alpha \in B} E_\alpha\right) \\ = \sum_{\alpha \in B} \mu(E_\alpha) \\ = \sum_{\alpha \in A} \mu(E_\alpha) \end{aligned}$$

and subtracting we conclude

$$\sum_{\alpha\in A}\mu(E_\alpha)-\sum\alpha\in B\mu(E_\alpha)=\sum_{\alpha\in A\setminus B}\mu(E_\alpha)=0,$$

but $\mu(\bigcup_{\alpha\in A\setminus B}E_\alpha)=M>0$. So we may assume without loss of generality that we have a family of sets $\{E_\alpha\}_{\alpha\in A}$ with $|A|\prec\kappa$, $\mu(E_\alpha)=0$ but $\mu(\cup_{\alpha\in A}E_\alpha)=M>0$. With this family we may define a measure ν on A by:

$$\nu(B)=\mu\left(\bigcup_{\alpha\in B}E_\alpha\right).$$

ν is finite since μ is, ν is non-trivial since for each $\alpha\in A$ $\nu(\{\alpha\})=\mu(E_\alpha)=0$, and ν is σ-additive. Indeed, if $\{F_n\}_{n=1}^\infty$ is a countable family of pairwise disjoint subsets of A then:

$$\begin{aligned}\sum_{n=1}^\infty\nu(F_n)&=\sum_{n=1}^\infty\mu\left(\bigcup_{\alpha\in F_n}E_\alpha\right)\\&=\mu\left(\bigcup_{n=1}^\infty\bigcup_{\alpha\in F_n}E_\alpha\right)\text{ since }\mu\text{ is }\sigma\text{-additive}\\&=\nu\left(\bigcup_{n=1}^\infty F_n\right),\end{aligned}$$

but $|A|\prec\kappa$, contradicting the minimality of κ. □

Since the smallest cardinal with a non-trivial finite measure κ induces a κ-additive measure on all larger cardinals, larger cardinals λ will only be of continued interest if they admit λ-additive measures. In light of this and Theorem 4.1 we define the following.

Definition 4.1. A cardinal $\kappa\succ\omega$ is *measurable* if κ admits a non-trivial κ-additive two-valued measure, and *real-valued measurable* if κ has a non-trivial finite κ-additive measure.

Remark 4.3: There is a non-trivial ω-additive (that is, finitely additive) two-valued measure on ω. Thus $\kappa\succ\omega$ is required, or we will have to deal with awkwardness either elsewhere in the definition or in proofs. As is standard in set theory we instead exclude ω from the definition.

We first note that (real-valued) measurable cardinals are *regular* in the sense that they are not the union of $\lambda\prec\kappa$ sets, each with cardinality strictly less than κ. (Note that regularity is the first condition of inaccessibility.)

Lemma 4.2. If κ is (real-valued) measurable, then κ is regular.

Proof. Let μ be a non-trivial κ-additive two valued measure on κ. If $X=\cup_{\alpha\in A}E_\alpha\subseteq\kappa$, with $|E_\alpha|,|A|\prec\kappa$, then since singletons have measure zero, by κ-additivity each E_α has measure zero, so X has measure zero by κ-additivity, hence $X\neq\kappa$. □

With this observation we can construct a rich measure on a measurable cardinal, further motivating our restriction to two-valued measures.

Theorem 4.4. If κ is measurable, then κ has a measure that takes on every value in $[0,1]$.

Proof. We first find a countable family $\{A_n\}$ of disjoint subsets of κ such that $|A_n| = \kappa$. Since κ is a set of ordinals we define an equivalence relation $\asymp$ on κ by: $\alpha \asymp \beta$ if and only if $\beta = \alpha + n$ or $\alpha = \beta + n$ for some $n \in \omega$. Then we can partition κ into equivalence classes. Let E be an equivalence class, since E is a set of ordinals there is a least ordinal $\alpha \in E$. Then $E = \{\alpha + n\}_{n\in\omega}$, so each equivalence class is countable. Since κ is regular, there must be κ equivalence classes. Let $\{\alpha_\lambda\}_{\lambda\in\kappa}$ be the set of least ordinals in each equivalence class. Then set $A_n = \{\alpha_\lambda + n\}_{\lambda\in\kappa}$. By construction $|A_n| = \kappa$ and each A_n is disjoint.

Thus the non-trivial two-valued κ-additive measure μ on κ induces measures μ_n on each A_n. It is then easily seen

$$\nu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(A_n \cap E).$$

is the desired measure on κ. □

We are now ready to answer the question posed in Section 3 and show that in ZFC we cannot prove the existence of a measurable cardinal. (We will make further remarks regarding real-valued measures later.)

Theorem 4.5. If κ is measurable, κ is inaccessible.

Proof. We have already seen (Lemma 4.2) that κ is not the sum of less than κ cardinals all smaller than κ. It remains to show that if $\lambda \prec \kappa$, then $2^\lambda \prec \kappa$.

To do this we will show that if for some $\lambda \prec \kappa$ we have $2^\lambda \succeq \kappa$ then a κ-additive measure on κ must be trivial. Recall that ${}^\lambda 2$, the space of functions from λ to $\{0, 1\}$ has cardinality 2^λ, so if ν is a κ-additive two valued measure on κ it induces a measure μ on ${}^\lambda 2$. We define a function $f \in {}^\lambda 2$ by transfinite induction so that $\{f\}$ must have measure 1. To do this, for an ordinal $\beta \le \lambda$ let $U(f, \beta) = \{g \in {}^\lambda 2 | f(\alpha) = g(\alpha), \alpha < \beta\}$. Note that for any ordinal β we can partition $U(f, \beta)$:

$$U(f,\beta) = \{\, g \in {}^\lambda 2 \,|\, f(\alpha) = g(\alpha), \alpha < \beta, g(\beta) = 0 \,\} \cup \{\, g \in {}^\lambda 2 \,|\, f(\alpha) = g(\alpha), \alpha < \beta, g(\beta) = 1 \,\}.$$

Note that $U(f, \emptyset) = {}^\lambda 2$. Since $\mu(U(f, \emptyset)) = 1$ for any f one of the two sets in the disjoint union above must have measure one. Call the two sets in the partition $U^0(f, \beta), U^1(f, \beta)$ respectively. Set $f(\emptyset) = i$ if and only if $\mu(U^i(f, \emptyset)) = 1$.

Suppose α is a successor ordinal and $\mu(U(f, \beta)) = 1$ for all $\beta \le \alpha$. Then, since μ is two valued and $\mu(U(f, \alpha)) = 1$ we must have that one of $U^0(f, \alpha), U^1(f, \alpha)$ has measure 1, we set $f(\alpha) = i$ if and only if $\mu(U^i(f, \alpha)) = 1$. Then f is defined for all ordinals less than or equal to α and $\mu(U(f, \alpha + 1)) = 1$.

If β is a limit ordinal and $\mu(U(f, \alpha)) = 1$ for all $\alpha < \beta$, define $f(\beta)$ as before, and note that:

$$U(f, \beta) = \cap_{\alpha<\beta} U(f, \alpha).$$

Using transfinite induction and κ-additivity we can extend continuity from above and below to the transfinite case, thus $\mu(U(f, \beta)) = 1$. So by transfinite induction f is defined as a function on λ taking values in $\{0, 1\}$, $U(f, \lambda) = \{f\}$ and $\mu(U(f, \lambda)) = 1$, i.e. the measure ν must be trivial since h is an injection; a contradiction. Hence κ is inaccessible. □

It follows from our previous discussion of inaccessible cardinals (Section 2.4) that we cannot hope to prove, using ZFC, that there are measurable cardinals. We have not excluded the possibility of proving the existence of a real-valued measurable cardinal in ZFC. It can be shown that if κ is real-valued measurable then for any $\lambda \prec \kappa$ we have that $\aleph(\lambda) \prec \kappa$, this is known as being *weakly inaccessible* [Dra74]. If we assume the Generalized Continuum Hypothesis (GCH), that $2^{\aleph_\alpha} = \aleph_{\alpha+1} = \aleph(\aleph_\alpha)$, then the notions of weakly inaccessible and inaccessible coincide, and so ZFC+GCH cannot prove the existence of real-valued measurable cardinals.

5 Consistency of the existence of accessible measurable cardinals

Solovay has shown that there is a model of ZF in which every subset of reals is Lebesgue measurable; this naturally induces a real-valued measure on 2^ω, which is accessible [Sol70]. Thus it is consistent with ZF that there is an accessible real-valued measurable cardinal. The proofs in the previous section made heavy use of the Axiom of Choice (totally ordering cardinals by $\preceq$, partitioning equivalence classes), these two facts motivate the question: for which consequences A of the Axiom of Choice that are weaker than choice (i.e. $ZF + A \not\vdash AC$) is ZF+A consistent with the existence of an accessible measurable cardinal?

The Axiom of Determinacy, proposed by Mycielski and Steinhaus [MS62], has some interesting consequences that help answer this question. We first state the axiom.

Definition 5.1. A *two-player ω-game with perfect information* is a game where two players alternate picking natural numbers forever, i.e. for each $n \in \omega$ there is an nth turn. This generates a sequence of natural numbers. The winner of the game is then decided by whether or not this sequence is in a set of sequences that win for player one, where the set is specified in advance and part of the game.

Definition 5.2. A two-player ω-game with perfect information A (A can be thought of as the set of winning sequences for player 1) is *determined* if player 1 has a winning strategy.

Axiom 10 (Determinacy). All two-player ω-games with perfect information are determined.

The Axiom of Determinacy is inconsistent with the Axiom of Choice, which makes its introduction somewhat controversial philosophically. However, it is a theorem in ZF+AD that $\aleph_1$ is measurable [Sol71]. ZF+AD also implies the following weak-choice form, known as countable choice [Bar89]:

Axiom 11 (Countable Choice).

$$\forall z(z \preceq \omega \wedge \forall x(x \in z \to \neg x = \emptyset \wedge \forall y(y \in z \to x \cap y = \emptyset \vee x = y)) \to \exists u \forall x \exists v(x \in z \to u \cap x = \{v\}))).$$

That is, for any countable family of sets we can perform the consequences of the axiom of choice. If we believe ZF+AD is consistent, then we must accept that ZF+CC is consistent with the existence of an accessible measurable cardinal. Checking Howard and Rubin's reference *Consequence of the Axiom of Choice* [HR91] quickly gives a list of choice-like principles that CC implies, giving further answers to our question. Are there principles stronger than Countable Choice consistent with the existence of an accessible measurable cardinal? This question appears to be open [Zwa], and it is beyond the scope of this paper to begin answering it.

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