EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACIAN

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ABSTRACT: The problem of determining the eigenvalues and eigenvectors for linear operators acting on finite dimensional vector spaces is a problem known to every student of linear algebra. This problem has a wide range of applications and is one of the main tools for dealing with such linear operators. Some of the results concerning these eigenvalues and eigenvectors can be extended to infinite dimensional vector spaces. In this article we will consider the eigenvalue problem for the Laplace operator acting on the L^2 space of functions on a bounded domain in $\mathbb{R}^n$. We prove that the eigenfunctions form an orthonormal basis for this space of functions and that the eigenvalues of these functions grow without bound.

1 NOTATION

In order to avoid confusion we begin by making explicit some notation that we will frequently use.

For a bounded domain $\Omega \subset \mathbb{R}^n$ we let $L^2(\Omega)$ be the usual real Hilbert space of real valued square integrable functions Ω, with inner product $\langle u, v\rangle_2 := \int_\Omega uv\,dx$ and norm $\|u\|_2 := \left(\int_\Omega u^2\,dx\right)^{1/2}$. We will also encounter the Sobolev space, denoted $H_0^{1,2}(\Omega)$, which is a similar space of real valued function with inner product and norm given instead by

$$\langle u, v\rangle_{1,2} = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx$$
$$\|u\|_{1,2} = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right)\,dx\right)^{1/2}.$$

Since this space is somewhat less common than $L^2(\Omega)$, the appendix reviews some elementary properties and theorems concerning this space which are useful in our analysis.

Our problem of interest in this article concerns the Laplace operator. This is a differential operator denoted by Δ and is given by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

where u is a sufficiently smooth real valued function, $u : \Omega \to \mathbb{R}$ and $x_1, x_2, \ldots x_n$ are the coordinates for $\Omega \subset \mathbb{R}^n$.

2 The Eigenvalue Problem

2.1 The eigenvalue equation

We consider the eigenvalue problem for the Laplacian on a bounded domain Ω. Namely, we look for pairs (λ, u) consisting of a real number λ called an *eigenvalue* of the Laplacian and a function $u \in C^2(\Omega)$ called an *eigenfunction* so that the following condition is satisfied

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Such eigenvalue/eigenfunction pairs have some very nice properties, some of which we will explore here. One fact of particular interest is that they form an orthonormal basis for $L^2(\Omega)$. This is an important and useful result to which we will work towards in this article.

Firstly, we will focus our attention to a weaker version of Equation 2.1. That is, we will examine a condition that is a necessary, but not sufficient, consequence of Equation 2.1. In particular, we will look for solutions u in the *Sobolev space* $H_0^{1,2}(\Omega)$ that obey the following equation for all test functions $v \in H_0^{1,2}(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx. \tag{2.2}$$

The following proposition shows that this condition is indeed weaker than Equation 2.1.

Proposition 2.1. If $u \in C^2(\Omega)$ satisfies Equation 2.1 then Equation 2.2 is satisfied too.

Proof. Suppose u is a twice differentiable function $u \in C^2(\Omega)$ that satisfies Equation 2.1. Given any $v \in H_0^{1,2}(\Omega)$, by definition of $H_0^{1,2}(\Omega)$ (see Appendix A), there is a sequence $v_k \in C_0^1(\Omega)$ so that $v_k \to v$ in the $H_0^{1,2}$ norm. We have that for any v_k

$$\begin{aligned} \int_\Omega (\Delta u + \lambda u) v_k \, dx = 0 \\ \int_\Omega (\Delta u) v_k \, dx = -\lambda \int_\Omega u v_k \, dx \\ -\int_\Omega \nabla u \cdot \nabla v_k \, dx = -\lambda \int_\Omega u v_k \, dx, \end{aligned} \tag{2.3}$$

where the last swap of derivatives is justified by the divergence theorem applied to the vector field $v_k \nabla u$ and utilizing the fact that $v_k \in C_0^1(\Omega)$ is compactly supported and so v_k vanishes on the boundary $\partial\Omega$. By definition of the norm on $H_0^{1,2}(\Omega)$ we have that for any $f \in H_0^{1,2}(\Omega)$ that $\|f\|_2 \le \|f\|_{1,2}$ and $\|\nabla f\|_2 \le \|\nabla f\|_{1,2}$ which means that since $v_k \to v$ in $H_0^{1,2}(\Omega)$ we automatically have that $v_k \to v$ and $\nabla v_k \to \nabla v$ in $L^2(\Omega)$. In particular, $\langle u, v_k \rangle_2 \to \langle u, v \rangle_2$ and $\langle \nabla u, \nabla v_k \rangle_2 \to \langle \nabla u, \nabla v \rangle_2$. Taking the limit as $k \to \infty$ of the equality in Equation 2.3 and using these limits gives us precisely Equation 2.2 as desired. □

Remark 2.2: Even more interesting perhaps is that the converse also holds. The weak functions $u \in H_0^{1,2}(\Omega)$ that satisfy Equation 2.2 can be shown, via some regularity results, to be smooth functions in $C^\infty(\Omega)$ and will also solve the original eigenvalue problem [McO03]. The proof of these regularity results is technical and would lead us too far from the eigenvalue problem which we investigate here, so we will content ourselves to simply proving results about the eigenfunctions that solve the weak equation, Equation 2.2, in this article.

The advantage of passing from the usual eigenvalue problem, Equation 2.1, to this weak equation is that we have moved from smooth functions to the Sobolev space $H_0^{1,2}(\Omega)$. In this restricted space, we can utilize certain results that would not hold in general and will be crucial to our analysis. The main tool we gain in this space is the Rellich compactness theorem, which allows us to find convergent subsequences of bounded sequences in $H_0^{1,2}(\Omega)$. Without this powerful tool, it would be impossible to prove the results which we strive for. For this reason, we will use Equation 2.2 as our defining equation rather than Equation 2.1. From now on when we refer to "eigenfunctions" or "eigenvalues" we mean solutions in $H_0^{1,2}(\Omega)$ of Equation 2.2 (rather than solutions of Equation 2.1). We will also refer to Equation 2.2 as "*the eigenvalue equation*" to remind ourselves of its importance.

Lemma 2.1. If u_1 and u_2 are eigenfunctions with eigenvalues λ_1 and λ_2 respectively and if $\lambda_1 \neq \lambda_2$ then $\langle u_1, u_2 \rangle_2 = 0$ and moreover $\langle \nabla u_1, \nabla u_2 \rangle_2 = 0$

Proof. Since u_1 and u_2 are both eigenfunctions, they satisfy the eigenvalue equation by definition. Plugging in $v = u_2$ into the eigenvalue equation for u_1 and $v = u_1$ into the eigenvalue equation for u_2 gives

$$\int_\Omega \nabla u_1 \cdot \nabla u_2 \, dx = \lambda_1 \int_\Omega u_1 u_2 \, dx$$
$$\int_\Omega \nabla u_2 \cdot \nabla u_1 \, dx = \lambda_2 \int_\Omega u_2 u_1 \, dx.$$

Subtracting the second equations from the first gives

$$(\lambda_1 - \lambda_2) \int_\Omega u_2 u_1 \, dx = 0,$$

so the condition $\lambda_1 \neq \lambda_2$ allows us to cancel out $\lambda_1 - \lambda_2$ to conclude $\int_\Omega u_2 u_1 = \langle u_1, u_2 \rangle_2 = 0$ as desired. Finally, notice that $\langle \nabla u_1, \nabla u_2 \rangle = \int_\Omega \nabla u_1 \cdot \nabla u_2 \, dx = \lambda_1 \int_\Omega u_1 u_2 \, dx = 0$ too. □

2.2 Constrained optimization and the Rayleigh quotient

Consider now the functionals from $H_0^{1,2}(\Omega) \to \mathbb{R}$

$$F(u) = \int_\Omega |\nabla u|^2 \, dx = \|\nabla u\|_2^2$$
$$G(u) = \int_\Omega u^2 \, dx - 1 = \|u\|_2^2 - 1.$$

These functionals have an intimate relationship with the eigenvalue problem. The following results makes this precise.

Lemma 2.2. If $u \in H_0^{1,2}(\Omega)$ is a local extremum of the functional F subject to the condition $G(u) = 0$, then u is an eigenfunction with eigenvalue $\lambda = F(u)$.

Proof. The proof of this relies on the Lagrange multiplier theorem in the calculus of variations setting (this result is exactly analogous to the usual Lagrange multiplier theorem on $\mathbb{R}^n$ with the first variation playing the role of the gradient). The Lagrange multiplier theorem states that if F and G are C^1-functionals on a Banach space X, and if $x \in X$ is a local extremum for the functional F subject to the condition that $G(x) = 0$ then either $\delta G(x)y = 0$ for all $y \in X$ or there exists some $\lambda \in \mathbb{R}$ so that $\delta F(x)y = \lambda \delta G(x)y$ for all $y \in X$. (Here $\delta F(u)v$ denotes the first variation of the functional F at the point u and in the direction of v.)

We use this theorem with the space $H_0^{1,2}(\Omega)$ serving the role of our Banach space, and F, G as defined above playing the role of the functionals under consideration. The first variation of F and G are easily computed

$$\begin{aligned}
\delta F(u)v &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F(u+\epsilon v)-F(u)\right)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_\Omega |\nabla u+\epsilon\nabla v|^2\,dx-\int_\Omega|\nabla u|^2\,dx\right)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_\Omega |\nabla u|^2+2\epsilon\nabla u\cdot\nabla v+\epsilon^2|\nabla v|^2-|\nabla u|^2\,dx\right)\\
&= \lim_{\epsilon\to 0}\left(\int_\Omega 2\,\nabla u\cdot\nabla v+\epsilon|\nabla v|^2\,dx\right)\\
&= 2\int_\Omega \nabla u\cdot\nabla v\,dx = 2\left\langle\nabla u,\nabla v\right\rangle_2.
\end{aligned}$$

A similar calculation yields

$$\delta G(u)v = 2\int_\Omega uv\;dx = 2\left\langle u,v\right\rangle_2.$$

Notice that $\delta G(u)u = 2\left\langle u,u\right\rangle_2 = 2\|u\|_2 = 2$ by the constraint $G(u)=0$. This means that $\delta G(u)v$ is not identically zero for all $v\in H_0^{1,2}(\Omega)$. Hence, since u is given to be a local extremum of F subject to $G(u)=0$ and $\delta G(u)$ is not identically zero, the Lagrange multiplier theorem tells us that there exists a λ so that for all $v\in H_0^{1,2}(\Omega)$ we have

$$\begin{aligned}
\delta F(u)v &= \lambda\delta G(u)v\\
2\left\langle\nabla u,\nabla v\right\rangle_2 &= 2\lambda\left\langle u,v\right\rangle_2.
\end{aligned}$$

Cancelling out the constant of 2 from both sides leaves us with exactly the eigenvalue equation! Hence u is an eigenfunction of eigenvalue λ as desired. Moreover, we can calculate λ directly using the fact that the above holds for all $v\in H_0^{1,2}(\Omega)$:

$$\begin{aligned}
F(u) &= \left\langle\nabla u,\nabla u\right\rangle_2\\
&= \lambda\left\langle u,u\right\rangle_2\\
&= \lambda,
\end{aligned}$$

where we have used $\left\langle u,u\right\rangle_2 = G(u)+1 = 1$ since $G(u)=0$ is given. □

Theorem 2.3. There exists some $u\in H_0^{1,2}(\Omega)$ so that u is a global minimum for F subject to the constraint $G(u)=0$.

Proof. Let us denote by $\mathcal{C}$ the constraint set we are working on, namely $\mathcal{C}=\{u\in H_0^{1,2}(\Omega): G(u)=0\}$. Notice that $G(u)=0$ precisely when $\|u\|_2=1$ so $\mathcal{C}$ is the set of unit norm functions. Let $I=\inf\{F(u): u\in\mathcal{C}\}$ be the infimum of F taken over this constraint set. We will prove that this infimum is actually achieved at some point $u\in\mathcal{C}$. By the definition of an infimum, we can find a sequence $\{u_j\}_{j=1}^\infty\subset\mathcal{C}$ so that $F(u_j)\le I+\frac{1}{j}$. In particular then, $\lim_{j\to\infty}F(u_j)=I$ and we also have that $F(u_j)=\|\nabla u_j\|_2^2\le I+1$ for all $j\in\mathbb{N}$. By the Poincaré inequality (Theorem A.1) we have then that $\|u_j\|_2\le C\|\nabla u_j\|_2\le C(I+1)$ for some constant C. Adding these inequalities together we see that

$$\begin{aligned}
\|u_j\|_{1,2}^2 &= \int_\Omega |\nabla u_j|^2+u_j^2\,dx\\
&= \|\nabla u_j\|_2^2+\|u_j\|_2^2\\
&\le (I+1)^2+C^2(I+1)^2\\
&< \infty.
\end{aligned}$$

In particular, this shows that u_j is a *bounded* sequence in $H_0^{1,2}(\Omega)$. Calling upon the Rellich compactness theorem (Theorem A.2), we know that we can find a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ of $\{u_j\}_{j=1}^{\infty}$ that converges in the L^2 sense to some element $\overline{u} \in \overline{\{u_j\}_{j=1}^{\infty}} \subset L^2(\Omega)$. Moreover, since $H_0^{1,2}(\Omega)$ is a Hilbert space, every bounded sequence contains a weak-convergent subsequence that converges in the weak topology on $H_0^{1,2}(\Omega)$. (It is a fact from the theory of functional analysis that the existence of such weak-convergent subsequences in a Banach space is equivalent to that Banach space being reflexive. As Hilbert spaces are self-dual by the Riesz representation theorem, they are certainly reflexive and hence we can always find such subsequences.) Hence, we may find a subsequence of $\{u_{j_k}\}_{k=1}^{\infty}$ that converges in the weak topology of $H_0^{1,2}(\Omega)$ to some $\overline{u}' \in H_0^{1,2}(\Omega)$ (for notational ease, we will continue to denote this subsequence by $\{u_{j_k}\}_{k=1}^{\infty}$). Of course, this subsequence still converges to $\overline{u}$ in $L^2(\Omega)$. Since $u_{j_k} \to \overline{u}$ in $L^2(\Omega)$, it follows that $\overline{u} = \overline{u}'$ i.e. we have that $u_{j_k} \to \overline{u}$ in the weak topology on $H_0^{1,2}(\Omega)$. This allows us to prove the following claim.

Claim. $\|\overline{u}\|_{1,2} \leq \liminf_{k\to\infty} \|u_{j_k}\|_{1,2}$

Proof of claim. Since $u_{j_k} \to \overline{u}$ in the weak topology on $H_0^{1,2}(\Omega)$, we have

$$\begin{aligned}
\|\overline{u}\|_{1,2}^2 &= \langle \overline{u}, \overline{u} \rangle_{1,2} \\
&= \lim_{k\to\infty} \langle \overline{u}, u_{j_k} \rangle_{1,2} \\
&= \liminf_{k\to\infty} \langle \overline{u}, u_{j_k} \rangle_{1,2} \\
&\leq \liminf_{k\to\infty} \|\overline{u}\|_{1,2} \|u_{j_k}\|_{1,2} \\
&= \|\overline{u}\|_{1,2} \liminf_{k\to\infty} \|u_{j_k}\|_{1,2}.
\end{aligned}$$

Cancelling out $\|\overline{u}\|_{1,2}$ from both sides yields the desired result. □

Using the above inequality and the fact that $\|\overline{u}\|_2 = \lim_{k\to\infty} \|u_{j_k}\|_2 = 1$ since $u_{j_k} \to \overline{u}$ in $L^2(\Omega)$, we can compute

$$\begin{aligned}
F(\overline{u}) &= \int_\Omega |\nabla \overline{u}|^2 \, dx \\
&= \int_\Omega \left(|\nabla \overline{u}|^2 + \overline{u}^2\right) \, dx - \int_\Omega \overline{u}^2 \, dx \\
&= \|u\|_{1,2}^2 - \|u\|_2^2 \\
&\leq \liminf_{k\to\infty} \|u_{j_k}\|_{1,2}^2 - \lim \|u_{j_k}\|_2^2 \\
&= \liminf_{k\to\infty} \left(\|u_{j_k}\|_{1,2}^2 - \|u_{j_k}\|_2^2\right) \\
&= \liminf_{k\to\infty} \left(\int_\Omega \left(|\nabla u_{j_k}|^2 + {u_{j_k}}^2\right) \, dx - \int_\Omega {u_{j_k}}^2 \, dx \right) \\
&= \liminf_{k\to\infty} \left(\int_\Omega |\nabla u_{j_k}|^2 \, dx \right) \\
&= \liminf_{k\to\infty} F(u_{j_k}) \\
&\leq \liminf_{k\to\infty} \left(I + \frac{1}{j_k} \right) \\
&\leq I,
\end{aligned}$$

but now, since $\|\overline{u}\|_2 = 1$, we have $\overline{u} \in \mathcal{C}$ so we have $F(\overline{u}) \geq I = \inf\{F(u) : u \in (C)\}$. Hence, combining the inequalities, we see that $F(\overline{u}) = I$ achieves the minimum for F restricted to $\mathcal{C}$ as desired. □

Remark 2.4: Theorem 2.3 shows that $\overline{u}$ is a global minimum of F subject to $G(u) = 0$. In particular then, it is a *local* extremum for F subject to $G(u) = 0$ so applying the result of Lemma 2.2 informs us that $\overline{u}$ is an eigenfunction with eigenvalue $\lambda = F(\overline{u})$. Since this is the smallest possible value of F subject to $G(u) = 0$, this is the smallest possible eigenvalue one could obtain. For this reason we shall call this eigenvalue λ_1 and the associated eigenfunction u_1.

Remark 2.5: By the definition of F, we notice that for any $u \in H_0^{1,2}(\Omega)$ and any scalar $c \in \mathbb{R}$, we have $F(cu) = c^2 F(u)$. This almost-linearity for scalars means that we can remove the condition $G(u) = 0$ from consideration in some sense by normalizing F by $\|u\|_2$. Notice that

$$\begin{aligned} \frac{F(u)}{\|u\|_2^2} &= \frac{\int_\Omega |\nabla u|^2 \, dx}{\|u\|_2^2} \\ &= \int_\Omega \left| \frac{\nabla u}{\|u\|} \right|^2 dx \\ &= F\left(\frac{u}{\|u\|} \right). \end{aligned}$$

Hence, minimizing $F(u)$ subject to $\|u\| = 1$ is the same as minimizing the quotient $\frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}$ with u running in all of $H_0^{1,2}(\Omega)$. This quotient is known as the *Rayleigh quotient*. This gives us a more notationally concise way to write down our smallest eigenvalue

$$\lambda_1 = \inf_{u \in H_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}.$$

2.3 The sequence of eigenvalues

To find the next eigenvalue, we can do something very similar. We first notice that the second smallest eigenvalue will have an eigenfunction that is orthogonal to u_1 by the result of Lemma 1, so we can restrict the search for this eigenfunction to the subspace $X_1 = \text{span}\{u_1\}^\perp = \{u \in H_0^{1,2}(\Omega) : \langle u, u_1 \rangle_2 = 0\}$. Since this is the null space of the continuous operator $\langle \cdot, u_1 \rangle_2$, this is a closed subspace of $H_0^{1,2}(\Omega)$ and hence can be thought of as a Hilbert space in its own right. By modifying the proof of Lemma 2 slightly by using X_1 as our Banach space rather than all of $H_0^{1,2}(\Omega)$, we see that any $u \in X_1$ that is a local extrema for F subject to $G(u) = 0$ will be an eigenfunction of eigenvalue $\lambda = F(u)$. By modifying the argument of Theorem 1 slightly by changing the restriction set $\mathcal{C}$ to be $\mathcal{C} = \{u \in X_1 : G(u) = 0$, the identical argument shows that there is some $u \in \mathcal{C}$ that achieves the minimum for F on this restricted set. This will be an extremum for F on X_1 subject to the restriction $G(u) = 0$, so by modified Lemma 2 this will be an eigenfunction, call it u_2. By arguments similar to the above, we find the associated eigenvalue λ_2 is

$$\begin{aligned} \lambda_2 &= \min\{F(u) : u \in \mathcal{C} \ \subset X_1\} \\ &= \inf_{u \in X_1} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}. \end{aligned}$$

Since $X_1 \subset H_0^{1,2}(\Omega)$, the Rayleigh quotient definition above tells us immediately that $\lambda_1 \leq \lambda_2$. Repeating this same idea inductively, we can define $X_n = \text{span}\{u_1, u_2, \ldots, u_n\}^\perp = \{u \in H_0^{1,2}(\Omega) : \langle u, u_i \rangle_2 = 0 \forall i \in 1, \ldots, n\}$ and by appropriately modifying Lemma 2.2 and Theorem 2.3 we will be able to justify the fact that the nth eigenvalue can be found by

$$\lambda_n = \inf_{u \in X_n} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Moreover, we can always find a normalized eigenfunction u_n that achieves this lower bound. Since $H_0^{1,2}(\Omega) \supset X_1 \supset X_2 \ldots$, we can see that this generates a sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ and eigenfunction $u_1, u_2, u_3, \ldots$ which are generated in such a way that they are all mutually orthogonal with respect to the $L^2(\Omega)$ inner product (our construction via the Rayleigh quotient restricted to X_n automatically orthogonalizes the eigenspaces of the degenerate eigenvalues). Moreover, these eigenfunctions have been normalized so that $\|u_n\|_2 = 1$ and also, by invoking the result of Lemma 2.1, we have then that $\|\nabla u_n\|_2 = \lambda_n \|u_n\|_2 = \lambda_n$. The following theorem shows that these eigenvalues tend to infinity.

Theorem 2.6. $\lim_{n\to\infty} \lambda_n = \infty$

Proof. This is another result that follows with the help of the Rellich compactness theorem. Since the sequence λ_i is non-decreasing, the only way that they could not tend to infinity is if they are bounded above. Suppose by contradiction that there is some constant M so that $\lambda_n < M$ for all $n \in \mathbb{N}$. Notice then that

$$\begin{aligned}\|\nabla u_n\|_2^2 &= \int_\Omega \nabla u_n \cdot \nabla u_n \; dx \\ &= \lambda_n \int_\Omega u_n^2 \; dx \\ &= \lambda_n \\ &\leq M,\end{aligned}$$

where we have used the eigenvalue equation with $v = u_n$ and the fact that $\|u_n\|_2 = 1$. Notice now that the sequence of eigenfunctions is *bounded* in $H_0^{1,2}(\Omega)$ since

$$\begin{aligned}\|u_n\|_{1,2}^2 &= \int_\Omega |\nabla u_n|^2 + u_n^2 \; dx \\ &= \|\nabla u_n\|_2^2 + \|u_n\|_2^2 \\ &\leq M + 1.\end{aligned}$$

By the Rellich compactness theorem, we can find a convergent subsequence u_{n_k} converging to some element of $L^2(\Omega)$. This subsequence, being convergent, is an L^2-Cauchy sequence, meaning in particular that $\|u_{n_k} - u_{n_{k+1}}\|_2^2 \to 0$ as $n \to \infty$. But orthonormality of u_n prohibits this as we have

$$\begin{aligned}\|u_{n_k} - u_{n_{k+1}}\|_2^2 &= \|u_{n_k}\|_2^2 - 2\left\langle u_{n_k}, u_{n_{k+1}}\right\rangle + \|u_{n_{k+1}}\|_2^2 \\ &= 1 - 0 + 1 \\ &> 0.\end{aligned}$$

This contradiction shows that our original assumption that the eigenvalues are bounded above by some M is impossible. Since the eigenvalues are nondecreasing, this is enough to show $\lim_{n\to\infty} \lambda_n = \infty$, as desired. □

2.4 Orthonormal basis

Finally, we have the machinery to prove that the eigenfunctions are not only an orthonormal set in $L^2(\Omega)$, but they are are a maximal orthonormal set: an orthonormal basis for $L^2(\Omega)$.

Theorem 2.7. For any $f \in L^2(\Omega)$, we can write $f = \sum_{n=1}^{\infty} \alpha_n u_n$ where $\alpha_n = \langle f, u_n \rangle_2$, where this infinite sum converges to f in the $L^2(\Omega)$ norm.

Proof. We first prove the result for functions $f \in H_0^{1,2}(\Omega)$ so that we may freely consider the (weak) derivative of f. Since $H_0^{1,2}(\Omega)$ is dense in $L^2(\Omega)$, this result can be extended to apply to any function $f \in L^2(\Omega)$. Given any $f \in H_0^{1,2}(\Omega)$, let ρ_N be the N-th error term between f and the partial sum $\sum_{n=1}^N \alpha_n u_n$, namely $\rho_N = f - \sum_{n=1}^N \alpha_n u_n$. To show that this sum converges to f in $L^2(\Omega)$ is tantamount to showing that $\|\rho_N\|_2 \to 0$ as $N \to \infty$. Firstly notice that $\langle \nabla\rho_N, \nabla u_k\rangle_2 = 0$ for every $1 \le k \le N$ since

$$\begin{aligned}
\langle \nabla\rho_N, \nabla u_k\rangle_2 &= \left\langle \nabla f - \sum_{n=1}^N \alpha_n \nabla u_n, \nabla u_k \right\rangle_2 \\
&= \langle \nabla f, \nabla u_k\rangle_2 - \sum_{n=1}^N \alpha_n \langle \nabla u_n, \nabla u_k\rangle_2 \\
&= \lambda_k \langle f, u_k\rangle_2 - \sum_{n=1}^N \alpha_n \|\nabla u_n\|_2^2 \delta_{nk} \\
&= \lambda_k \alpha_k - \alpha_k \|\nabla u_k\|_2^2 \\
&= \lambda_k \alpha_k - \alpha_k \lambda_k \\
&= 0,
\end{aligned}$$

where we have used the eigenvalue equation with $v = f$ and the orthonormality of u_n. In a very similar way, we have that $\langle \rho_N, u_k\rangle_2 = 0$ for every $1 \le k \le N$ since

$$\begin{aligned}
\langle \rho_N, u_k\rangle_2 &= \left\langle f - \sum_{n=1}^N \alpha_n u_n, u_k \right\rangle_2 \\
&= \langle f, u_k\rangle_2 - \sum_{n=1}^N \alpha_n \langle u_n, u_k\rangle_2 \\
&= \alpha_k - \sum_{n=1}^N \alpha_n \delta_{nk} \\
&= 0.
\end{aligned}$$

Since this holds for all $1 \le k \le N$ we conclude that $\rho_N \in \operatorname{span}\{u_1, u_2, \dots, u_N\}^\perp = X_N$. We hence have the following inequality which follows from the Rayleigh quotient definition of λ_{N+1}

$$\begin{aligned}
\frac{\int_\Omega |\nabla\rho_N|^2\,dx}{\int_\Omega \rho_N^2\,dx} &\ge \inf_{u\in X_N} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx} \\
&= \lambda_{N+1},
\end{aligned}$$

and hence:

$$\|\nabla\rho_N\|_2^2 \ge \lambda_{N+1}\|\rho_N\|_2^2.$$

This inequality is the crux of the proof, for we see that

$$\begin{aligned}
\|\nabla f\|_2^2 &= \|\nabla\rho_N + \sum_{n=1}^N \alpha_n \nabla u_n\|_2^2 \\
&= \|\nabla\rho_N\|_2^2 + \|\sum_{n=1}^N \alpha_n \nabla u_n\|_2^2 \\
&\ge \lambda_{N+1}\|\rho_N\|_2^2 + 0,
\end{aligned}$$

where we have used the fact that $\langle \nabla \rho_N, \nabla u_k \rangle_2 = 0$ for every $1 \le k \le N$ to enable the Pythagorean theorem in the second equality. Now the fact that the $\lambda_{N+1} \to \infty$ forces $\|\rho_N\|_2 \to 0$ because otherwise, the right hand side of the equation diverges as $N \to \infty$, while the left hand side is independent of N and finite as $f \in H_0^{1,2}(\Omega)$, a contradiction. Hence $\|\rho_N\|_2 \to 0$ meaning that $\sum_{n=1}^{\infty} \alpha_n u_n$ converges to f in the $L^2(\Omega)$ sense, as desired.

To extend this result from functions $f \in H_0^{1,2}(\Omega)$ as above to more general $f \in L^2(\Omega)$ we use the fact that $H_0^{1,2}(\Omega)$ is dense in $L^2(\Omega)$. (This is not surprising since the even more restrictive set $C_0^\infty(\Omega)$ can be shown to be dense in $L^2(\Omega)$). Given any $f \in L^2(\Omega)$, we may find some family $\{f_\epsilon\} \subset H_0^{1,2}(\Omega)$ so that $f_\epsilon \to f$ in $L^2(\Omega)$ as $\epsilon \to 0$. In particular then, by the Cauchy Shwarz inequality, we have for each $n \in \mathbb{N}$ that $\langle f - f_\epsilon, u_n \rangle_2 \to 0$ as $\epsilon \to 0$ and hence $\alpha_{n,\epsilon} = \langle f_\epsilon, u_n \rangle \to \alpha_n = \langle f, u_n \rangle$ in this limit. By careful addition and subtraction by zero, and by use of the Minkowski inequality on $L^2(\Omega)$ we have

$$\left\| f - \sum_{n=1}^{N} \alpha_n u_n \right\|_2 \le \|f - f_\epsilon\|_2 + \left\| f_\epsilon - \sum_{n=1}^{N} \alpha_{n,\epsilon} u_n \right\|_2 + \left\| \sum_{n=1}^{N} \left(\alpha_{n,\epsilon} - \alpha_n\right) u_n \right\|_2 ,$$

but now by Bessel's inequality, which holds for any orthonormal set (such as the set u_n by their construction), applied to the function $f_\epsilon - f$, we have that

$$\begin{aligned} \left\| \sum_{n=1}^{N} \left(\alpha_{n,\epsilon} - \alpha_n\right) u_n \right\|_2 &\le \left\| \sum_{n=1}^{\infty} \left(\alpha_{n,\epsilon} - \alpha_n\right) u_n \right\|_2 \\ &= \left\| \sum_{n=1}^{\infty} \langle f_\epsilon - f, u_n \rangle_2 \right\| \\ &\le \|f - f_\epsilon\|_2, \end{aligned}$$

which is then added to first inequality to get

$$\left\| f - \sum_{n=1}^{N} \alpha_n u_n \right\|_2 \le 2\|f - f_\epsilon\|_2 + \left\| f_\epsilon - \sum_{n=1}^{N} \alpha_{n,\epsilon} u_n \right\|_2 .$$

By taking ϵ small enough so that $2\|f - f_\epsilon\|_2$ becomes arbitrarily small and N large enough so that $\left\| f_\epsilon - \sum_{n=1}^{N} \alpha_{n,\epsilon} u_n \right\|_2$ is arbitrarily small, we can bound $\left\| f - \sum_{n=1}^{N} \alpha_n u_n \right\|_2$ to be arbitrarily small as well, and hence the L^2 difference between f and its N-th partial eigenfunction expansion must vanish in the limit $N \to \infty$. This shows that any $f \in L^2(\Omega)$ can be written as $f = \sum_{n=1}^{\infty} \alpha_n u_n$ in the L^2 sense, where $\alpha_n = \langle f, u_n \rangle_2$, meaning that the eigenfunctions do indeed form an orthonormal basis for all of $L^2(\Omega)$. □

References

[Eva98] Lawrence C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[McO03] Robert C. McOwen, *Partial Differential Equations: Methods and Applications, Second Edition*, Prentice Hall, 2003.

A Sobolev spaces

In this appendix we will fill in some background concerning the simplest Sobolev space, $H_0^{1,2}(\Omega)$, which is used in our investigation of the eigenvalues/eigenfunction pairs above. We also prove the Poincaré inequality, which we call on in this analysis and we very roughly motivate the ideas in the proof of the Rellich compactness theorem which is in some ways the cornerstone of many of the results about eigenvalue/eigenfunction pairs.

A.1 The Sobolev space $H_0^{1,2}(\Omega)$

The Sobolev space $H_0^{1,2}(\Omega)$ is a refinement of $L^2(\Omega)$ whose additional structure is of some use to us. One defines $H_0^{1,2}(\Omega)$ by first defining a new inner product on the the set of continuously differentiable, compactly supported functions $C_0^1(\Omega)$, namely the inner product $\langle \cdot, \cdot \rangle_{1,2}$:

$$\langle u, v \rangle_{1,2} = \int_\Omega (\nabla u \cdot \nabla v + uv)\ dx.$$

The induced norm from this inner product is

$$\|u\|_{1,2} = \sqrt{\langle u, u \rangle_{1,2}} = \left(\int_\Omega \left(|\nabla u|^2 + u^2 \right)\ dx \right)^{1/2}.$$

Just as $C_0^1(\Omega)$ is not complete in the usual norm $\langle \cdot, \cdot \rangle_2$ from $L^2(\Omega)$, $C_0^1(\Omega)$ with this norm is *not complete*. However, by the definition of this norm, any sequence $\{u_k\}_{k=1}^\infty$ which is Cauchy in the $\|\cdot\|_{1,2}$ norm will be Cauchy in the $L^2(\Omega)$ norm too. This is by virtue of the fact that $\|u_k - u_j\|_2 \le \|u_k - u_j\|_{1,2} \to 0$ since u_k is $\|\cdot\|_{1,2}$-Cauchy. (This inequality holds as the $H_0^{1,2}(\Omega)$ norm has an extra non-negative term $|\nabla u|^2$ in the integral, which gives a nonnegative contribution to this norm). Since $L^2(\Omega)$ is complete, we conclude that such a Cauchy sequence converges to some $u \in L^2(\Omega)$. By including all the limits of all the $\|\cdot\|_{1,2}$-Cauchy sequences, we get an honest Hilbert space which we denote by $H_0^{1,2}(\Omega)$, called the *Sobolev space*. In other words, the definition of this Sobolev space is

$$H_0^{1,2}(\Omega) = \overline{C_0^1(\Omega)}^{\|\cdot\|_{1,2}}.$$

This is the completion of $C_0^1(\Omega)$ with respect to the $\|\cdot\|_{1,2}$ norm. As remarked before, this completion consists of adding in some $L^2(\Omega)$ functions, and hence the resulting space is a subset of $L^2(\Omega)$.

A.2 Weak derivatives on $H_0^{1,2}(\Omega)$

Notice that by the above definition, the functions $u \in H_0^{1,2}(\Omega)$ do not necessarily have derivatives in the classical sense, but they do have *weak* derivatives defined by $\frac{\partial u}{\partial x_j} = \lim_{k\to\infty} \frac{\partial u_k}{\partial x_j}$ where u_k is any sequence in $C_0^1(\Omega)$ which converges to u in $L^2(\Omega)$. Notice that this is indeed the weak derivative since for any test function $v \in C_0^\infty(\Omega)$ we have that

$$\begin{aligned}
\int_\Omega (u - u_k) \left(-\frac{\partial v}{\partial x_j} \right)\ dx &= \left\langle u - u_k, -\frac{\partial v}{\partial x_j} \right\rangle_2 \\
&\le \|u - u_k\|_2 \left\| \frac{\partial v}{\partial x_j} \right\|_2 \\
&\to 0,
\end{aligned}$$

and hence we have that

$$\begin{aligned}
\int_\Omega u \left(-\frac{\partial v}{\partial x_j} \right)\ dx &= \int_\Omega \lim_{k\to\infty} u_k \left(-\frac{\partial v}{\partial x_j} \right)\ dx \\
&= \int_\Omega \left(\lim_{k\to\infty} \frac{\partial u_k}{\partial x_j} \right) v\, dx,
\end{aligned}$$

where the swap of derivatives is justified by the divergence theorem because both functions are at least $C_0^1(\Omega)$ and have compact support. Since this holds for any test function v, then u is the weak solution to $\frac{\partial u}{\partial x_j} = \lim_{k\to\infty} \frac{\partial u_k}{\partial x_j}$ and this is what we mean when we say the weak derivative of u exists and is equal to this limit.

A.3 The Poincaré inequality

Theorem A.1 (Poincaré Inequality). If Ω is a bounded domain, then there is a constant C depending only on Ω so that

$$\int_\Omega u^2\,dx \le C\int_\Omega |\nabla u|^2\,dx$$

for all $u \in C_0^1(\Omega)$ and by completion for all $u \in H_0^{1,2}(\Omega)$.

Proof. For $u \in C_0^1(\Omega)$, we find an $a \in \mathbb{R}$ large enough so that the cube $Q = \{x \in \mathbb{R}^n : |x_j| < a, 1 \le j \le n\}$ contains Ω. Performing an integration by parts in the x_1-direction then gives (the non-integral terms vanish since $u = 0$ on the boundary of Q)

$$\begin{aligned}\int_\Omega u^2\,dx &= \int_\Omega 1\cdot u^2\,dx \\ &= -\int_\Omega x_1\frac{\partial u^2}{\partial x_1}\,dx \\ &= -2\int_\Omega x_1 u\frac{\partial u}{\partial x_1}\,dx \\ &= 2a\int_\Omega |u||\frac{\partial u}{\partial x_1}|\,dx.\end{aligned}$$

Using the Cauchy-Schwarz inequality for $L^2(\Omega)$ now gives

$$\begin{aligned}\int_\Omega u^2\,dx &\le 2a\int_\Omega |u||\frac{\partial u}{\partial x_1}|\,dx \\ &\le 2a\|u\|_2\|\frac{\partial u}{\partial x_1}\|_2 \\ &\le 2a\|u\|_2\|\nabla u\|_2.\end{aligned}$$

Dividing through by $\|u\|_2$ gives the desired result with $C = (2a)^2$. For $u \in H_0^{1,2}(\Omega)$, we find a sequence $\{u_k\}_{k=1}^\infty \subset C_0^1(\Omega)$ converging to u in the $H_0^{1,2}(\Omega)$ norm (this is by definition of $H_0^{1,2}(\Omega)$). We have then that $\|u-u_j\|_2 \le \|u-u_j\|_{1,2} \to 0$ as $j\to\infty$ and similarly $\|\nabla u - \nabla u_j\|_2 \le \|u-u_j\|_{1,2} \to 0$. Hence, by making use of the Cauchy-Schwarz inequality, we have that $\|u_j\|_2 \to \|u\|_2$ and $\|\nabla u_j\|_2 \to \|\nabla u\|_2$ in the limit $j\to\infty$, which allows us to use the Poincaré inequality on $u_j \in C_0^1(\Omega)$ in the limit $j\to\infty$ to conclude that $\int_\Omega u^2\,dx \le C\int_\Omega |\nabla u|^2\,dx$ as desired. □

Theorem A.2 (Rellich Compactness). For a bounded domain Ω, the inclusion map $I : H_0^{1,2}(\Omega) \to L^2(\Omega)$ is a *compact* operator meaning that it takes bounded sets in $H_0^{1,2}(\Omega)$ to *totally bounded* sets (also known as precompact) in $L^2(\Omega)$. By the sequential compactness characterization of compact sets, this is equivalent to saying that for any bounded sequence $\{u_n\}_{n=1}^\infty \in H_0^{1,2}(\Omega)$, there is a subsequence $\{u_{n_k}\}_{k=1}^\infty$ that converges in the L^2 sense to some $u \in L^2(\Omega)$.

Proof sketch. To do in full detail, the proof is rather long and technical, so we will omit most of the details and instead sketch the main themes of the proof. Given any bounded sequence $\{f_n\}_{n=1}^{\infty} \subset H_0^{1,2}(\Omega)$, the idea is to first smooth out the sequence of functions by convolving them with a so-called mollifier function η_ϵ depending on a choice of ϵ so that the resultant sequence of smoothed (also called mollified) functions $\{\eta_\epsilon * f_n\}_{n=1}^{\infty}$ is better behaved than the original sequence $\{f_n\}_{n=1}^{\infty}$ is. By choosing η_ϵ appropriately, so that η_ϵ is bounded and with bounded derivative, one can verify that the resulting sequence of smoothed functions $\{\eta_\epsilon * f_n\}_{n=1}^{\infty}$ will also be bounded and with bounded derivative. This derivative bound is enough to see that this family is equicontinuous, so one can invoke the Arzela-Ascoli theorem to see that these smoothed functions have a uniformly convergent subsequence. Using the boundedness of $\{f_n\}_{n=1}^{\infty}$ in $H_0^{1,2}(\Omega)$ allows one to argue that as $\epsilon \to 0$, these mollified functions converge uniformly back to the original sequence of functions. Since the mollified functions have convergent subsequences and since the mollified functions return to the original sequence, a little more analysis allows one to verify that the original sequence will enjoy a convergent subsequence as well. □

Remark A.3: This theorem is sometimes filed under the title "The Kondrachov compactness theorem", after V. Kondrachov who generalized Franz Rellich's result in the more general compact map $H_0^{1,p}(\Omega)$ into $L^q(\Omega)$ whenever $1 \leq q \leq np/(n-p)$.