

# A SYSTEMATIC CONSTRUCTION OF ALMOST INTEGERS

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ABSTRACT: Motivated by the search for “almost integers”, we describe the algebraic integers known as Pisot numbers, and explain how they can be used to easily find irrational values that can be arbitrarily close to whole numbers. Some properties of the set of Pisot numbers are briefly discussed, as well as some applications of these numbers to other areas of mathematics.

## 1 INTRODUCTION

It is a curious occurrence when an expression that is known to be a non-integer ends up having a value surprisingly close to a whole number. Some examples of this phenomenon include:

$$\begin{aligned} e^\pi - \pi &= 19.9990999791\dots \\ \left(\frac{23}{9}\right)^5 &= 109.0000338701\dots \\ 88 \ln 89 &= 395.0000005364\dots \end{aligned}$$

These peculiar numbers are often referred to as “almost integers”, and there are many known examples. Almost integers have attracted considerable interest among recreational mathematicians, who not only try to generate elegant examples, but also try to justify the unusual behaviour of these numbers. In most cases, almost integers exist merely as numerical coincidences, where the value of some expression just happens to be very close to an integer. However, sometimes there actually is a clear, mathematical reason why certain irrational numbers *should* be very close to whole numbers. In this paper, we’ll look at the a set of numbers called the Pisot numbers, and how they can be used to systematically construct infinitely many examples of almost integers.

In Section 2, we will prove a result about powers of roots of polynomials, and use this as motivation to define the Pisot numbers. We will also show how Pisot numbers can generate many almost integers. In Section 3, we will explore the set  $S$  of Pisot numbers in more detail, and in Section 4, we will list some other properties and applications of the Pisot numbers. Finally, Section 5 will present some concluding remarks.

## 2 PISOT NUMBERS AND ALMOST INTEGERS

We begin by recalling some definitions from the study of polynomials.

*Definition 2.1.* A number is called an *algebraic integer* if it is the root of some polynomial of the form  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ , where the coefficients  $a_i$  are all integers. If  $f(x)$  is the minimal polynomial for some algebraic integer  $\alpha$ , then the roots of  $f(x)$  other than  $\alpha$  are called the *Galois conjugates* of  $\alpha$ .

The main result used to motivate the construction of Pisot numbers is that for any monic polynomial  $f(x)$ , the sum of the powers of the roots of  $f(x)$  will always be exactly an integer. This is made precise in the following theorem.

*Theorem 2.1.* Let  $f(x)$  be a monic, irreducible polynomial of degree  $d$  with (not necessarily distinct) roots  $\theta_1, \dots, \theta_d$ . Then  $\theta_1^n + \dots + \theta_d^n$  is an integer for all integers  $n \geq 0$ .

*Proof.* We may write  $f(x) = (x - \theta_1) \cdots (x - \theta_d)$ . Taking the natural logarithm of both sides gives

$$\log f(x) = \sum_{i=1}^d \log(x - \theta_i).$$

Differentiating, we obtain

$$\frac{d}{dx} \log f(x) = \frac{f'(x)}{f(x)} = \frac{1}{x - \theta_1} + \dots + \frac{1}{x - \theta_d}.$$

If we now substitute  $1/x$  for  $x$  in the above equation, then we get

$$\frac{f'(1/x)}{f(1/x)} = \frac{1}{1/x - \theta_1} + \dots + \frac{1}{1/x - \theta_d},$$

and so

$$\begin{aligned} \frac{x^{d-1} f'(1/x)}{x^d f(1/x)} &= \frac{1}{1 - x\theta_1} + \dots + \frac{1}{1 - x\theta_d} \\ &= \sum_{i=1}^d \frac{1}{1 - x\theta_i}. \end{aligned}$$

By expressing the ratio  $1/(1 - x\theta_i)$  as an infinite geometric series, this equation becomes

$$\begin{aligned} \frac{x^{d-1} f'(1/x)}{x^d f(1/x)} &= \sum_{i=1}^d \sum_{n=0}^{\infty} x^n \theta_i^n \\ &= \sum_{n=0}^{\infty} \left( \sum_{i=1}^d \theta_i^n \right) x^n \\ &= \sum_{n=0}^{\infty} t_n x^n, \end{aligned}$$

where we let

$$t_n = \sum_{i=1}^d \theta_i^n.$$

To prove the theorem, it remains to show that  $t_n \in \mathbb{Z}$  for all integers  $n \geq 0$ .

We do this by first writing  $f(x)$  in a different way. There exist integers  $a_0, \dots, a_{d-1}$  such that we may write

$$f(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + a_0$$

and so

$$f'(x) = dx^{d-1} + (d-1)a_{d-1}x^{d-2} + \dots + 2a_2x + a_1.$$

This gives us that

$$\begin{aligned} x^{d-1} f'(1/x) &= d + (d-1)a_{d-1}x + \dots + 2a_2x^{d-2} + a_1x^{d-1} \\ x^d f(1/x) &= 1 + a_{d-1}x + \dots + a_1x^{d-1} + a_0x^d. \end{aligned}$$

Putting all of this together, we get

$$\begin{aligned}\frac{x^{d-1}f'(1/x)}{x^d f(1/x)} &= \sum_{n=0}^{\infty} t_n x^n \\ x^{d-1}f'(1/x) &= (x^d f(1/x)) \sum_{n=0}^{\infty} t_n x^n \\ d + (d-1)a_{d-1}x + \cdots + a_1 x^{d-1} &= (1 + a_{d-1}x + \cdots + a_0 x^d) \sum_{n=0}^{\infty} t_n x^n.\end{aligned}$$

We now compare coefficients on both sides of this equation, to obtain the system

$$\begin{aligned}d &= t_0 \\ (d-1)a_{d-1} &= t_1 + a_{d-1}t_0 \\ (d-2)a_{d-2} &= t_2 + a_{d-1}t_1 + a_{d-2}t_2 \\ &\vdots\end{aligned}$$

from which we see that all of the  $t_i$  do, in fact, take integer values. Hence  $\theta_1^n + \cdots + \theta_d^n \in \mathbb{Z}$  for all integers  $n \geq 0$ , as desired.  $\square$

We now present the definition of a Pisot number. These numbers were first studied by Thue [Thu12] in 1912, and were later looked at by Hardy [Har19] in 1919. However, they only gained popularity in the wider mathematical community after Pisot's dissertation concerning them in 1938 [Pis38].<sup>1</sup>

*Definition 2.2.* A *Pisot number* is a real, algebraic integer larger than 1 whose Galois conjugates all have absolute value less than 1.

Pisot numbers can be identified by looking at the roots of their minimal polynomials on the complex plane. If  $f(x)$  is the minimal polynomial for a Pisot number  $\alpha$ , then all of the roots of  $f(x)$  lie strictly within the unit disc on the complex plane, except for  $\alpha$ , which lies outside the disc on the positive real axis.

Using Theorem 2.1, we will show that by taking high powers of Pisot numbers, we obtain values that are very close to whole numbers. First, however, we need to make more precise what we mean by ‘‘closeness to a whole number’’.

*Definition 2.3.* Given a real number  $x$ , we define the *distance (from  $x$ ) to the nearest integer* to be  $\|x\| = |x - n|$ , where  $n$  is taken to be the closest integer to  $x$ .

For example, we have  $\|7\| = 0$ ,  $\|\pi\| = 0.14159\dots$ , and  $\|2.6\| = 0.4$ . Note that for any real number  $x$ ,  $\|x\|$  ranges between 0 and 0.5, and  $\|x\| = 0$  if and only if  $x$  is an integer. Effectively,  $\|x\|$  measures how close  $x$  is to being a whole number, with small values of  $\|x\|$  corresponding to almost integers.

*Theorem 2.2.* If  $\alpha$  is a Pisot number, then  $\lim_{n \rightarrow \infty} \|\alpha^n\| = 0$ .

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<sup>1</sup>Pisot's preliminary results were independently proven in 1941 by Vijayaraghavan [Vij41], who was also interested in studying this class of numbers. Some mathematicians therefore refer to these numbers as *Pisot-Vijayaraghavan numbers*, or PV numbers, in recognition of the contributions of both mathematicians, as suggested by Salem in 1943. Unfortunately for Vijayaraghavan, however, most of the literature regarding these numbers in refers to them simply as Pisot numbers.

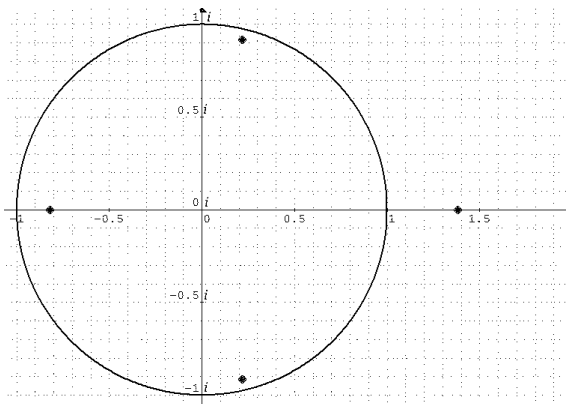


Figure 2.1: A plot of the roots of the polynomial  $f(x) = x^4 - x^3 - 1$  in the complex plane, along with the unit circle. Since only one root of the polynomial, approximately 1.38, lies outside the unit disc, this root is a Pisot number.

*Proof.* Let  $\alpha$  have Galois conjugates  $\theta_2, \dots, \theta_d$ . Since  $|\theta_i| < 1$  for all  $i = 2, \dots, d$ , we have

$$\lim_{n \rightarrow \infty} \theta_i^n = 0,$$

for all  $i = 2, \dots, d$ , and in particular,

$$\lim_{n \rightarrow \infty} (\theta_2^n + \theta_3^n + \dots + \theta_d^n) = 0.$$

From the result of Theorem 2.1, we have for any  $n \geq 0$ , there exists some integer  $b_n$  such that

$$\alpha^n + \theta_2^n + \dots + \theta_d^n = b_n.$$

We now see that

$$\begin{aligned} b_n - \alpha^n &= \theta_2^n + \dots + \theta_d^n \\ \lim_{n \rightarrow \infty} (b_n - \alpha^n) &= \lim_{n \rightarrow \infty} (\theta_2^n + \dots + \theta_d^n) \\ &= 0 \end{aligned}$$

Thus as  $n$  grows large,  $\alpha^n$  gets arbitrarily close to the integer  $b_n$ , and correspondingly, we have that  $\|\alpha^n\|$  approaches 0, as desired.  $\square$

At last, we see a systematic way of constructing almost integers. Given any Pisot number  $\alpha$ , by taking successively higher powers of  $\alpha$ , we obtain values that get closer and closer to whole numbers. The time is ripe for us to look at some examples of Pisot numbers, and the almost integers that they generate.

- For every integer  $n \geq 2$ , we have  $n$  as the root of  $f(x) = x - n$ , and so every positive integer larger than 1 is a Pisot number. The powers of integers, however, are exactly integers already, so these Pisot numbers are not too useful in generating almost integers.
- The golden ratio  $\phi = (1 + \sqrt{5})/2 = 1.61803\dots$  is the root of  $f(x) = x^2 - x - 1$ , with Galois conjugate  $-\phi^{-1} = -0.61803\dots$  having absolute value less than 1. Thus  $\phi$  is a Pisot number.
- $1 + \sqrt{2} = 2.41421\dots$  is a Pisot number, with minimal polynomial  $f(x) = x^2 - 2x - 1$  and  $1 - \sqrt{2} = -0.41421\dots$  as its Galois conjugate.
- $1.46557\dots$  is a cubic Pisot number, with minimal polynomial  $f(x) = x^3 - x^2 - 1$  and  $-0.23278\dots \pm i0.79255\dots$  as its Galois conjugates.

Table 2.1: Some Pisot numbers and corresponding almost integers.

$\alpha$	$1.61803\dots = \frac{1+\sqrt{5}}{2}$	$2.41421\dots = 1 + \sqrt{2}$	$1.46557\dots$
$\alpha^2$	2.6180...	5.8284...	2.1478...
$\alpha^3$	4.2360...	14.0710...	3.1478...
$\alpha^4$	6.8541...	33.97056...	4.6134...
$\alpha^5$	11.0901...	82.01219...	6.7613...
$\alpha^{10}$	122.99186...	6725.9998513...	45.7161...
$\alpha^{15}$	1364.000731...	551614.0000018128...	309.10353...
$\alpha^{20}$	15126.9999338...	45239073.99999997789...	2089.96315...
$\alpha^{25}$	167761.00000596...	3710155682.000000000269...	14131.01273...

Some of the almost integers corresponding to powers of these Pisot numbers are listed in Table 2.1.

It is apparent that by taking higher powers of these Pisot numbers, we obtain irrational numbers that get closer and closer to whole numbers. As can be deduced from Theorem 2.2, the rate at which these powers approach whole numbers depends on how large the absolute values of the Galois conjugates of the Pisot numbers are. We can see that the powers of  $1 + \sqrt{2}$  approach integers rapidly, since its Galois conjugate has a relatively small modulus of  $0.414\dots$ . On the other hand, the Galois conjugates of  $1.46557\dots$  both have absolute value  $0.82603\dots$ , which is much closer to 1, resulting in fairly slow convergence to almost integers.

One of Pisot's original and noteworthy results was that the Pisot numbers are the only algebraic numbers that can generate almost integers this way. In particular, he was able to show that if  $\alpha > 1$  is an algebraic number, then the existence of some nonzero, real  $\lambda$  such that

$$\lim_{n \rightarrow \infty} \|\lambda \alpha^n\| = 0$$

is sufficient to conclude that  $\alpha$  is a Pisot number [Pis38]. It is unknown if this result remains true for non-algebraic  $\alpha$ , as no transcendental counterexamples have been found.

### 3 THE STRUCTURE OF $S$

In the previous section, we have shown that a clever method to construct almost integers is to find a Pisot number and evaluate it at high powers. Only one detail was missing from our discussion; we have not described a process for obtaining a non-trivial Pisot number to begin with, or mentioned anything on the distribution of Pisot numbers. It turns out that the set of Pisot numbers, commonly referred to as  $S$ , has been studied in great detail, and is very well understood. In this section, we will look at some properties of the structure of this set.

#### THE SET $S$ IS COUNTABLY INFINITE.

We saw earlier that every natural number larger than 1 is a Pisot number, and hence the set of Pisot numbers must be infinite. On the other hand,  $S$  is a strict subset of the set of algebraic integers, which is known to be countable, so  $S$  is also a countable set.

THE SET  $S$  IS CLOSED.

Recall that a set is closed when it contains all of its limit points. The highly nontrivial fact that  $S$  is a closed subset of  $\mathbb{R}$  is due to a proof by Salem [Sal44],<sup>2</sup> who clarified that the set of Pisot numbers is not dense in  $\mathbb{R}$  (for if  $S$  were dense, then every point in  $\mathbb{R}$  would be either in  $S$  or a limit point of  $S$ ).

THE SMALLEST ELEMENT OF  $S$  IS KNOWN.

The set of Pisot numbers is unbounded from above, since every integer larger than 1 is an element of  $S$ . However, it is bounded from below, since Pisot numbers are all strictly larger than 1. Since  $S$  is closed, it must have a least element; Siegel [Sie44] proved that this smallest Pisot number is the positive root of  $x^3 - x - 1$ , which is approximately 1.32472.

THE SET  $S$  HAS INFINITELY MANY LIMIT POINTS.

It is known that the set of limit points of  $S$  has limit points of its own. In fact, more than that can be said. Let us define the *derived sets* of  $S$  as a sequence of sets  $S^{(0)}, S^{(1)}, S^{(2)}, \dots$  such that  $S^{(0)} = S$ , and for  $n \geq 1$ ,  $S^{(n+1)}$  is the set of limit points of  $S^{(n)}$ . Then it is known that  $S^{(n)}$  is nonempty for any finite  $n$ . The smallest element of  $S^{(2)}$  is known to be 2; that is, 2 is a limit point of limit points of  $S$  [Ber80]. It was determined by Bertin [Ber80], in fact, that  $n \in S^{(2n-2)}$  for all  $n$ . A consequence of this is that there are a huge amount of Pisot numbers clustered around the real line, particularly near the integers.

THE SUBSET  $S \cap [1, 2]$  IS COMPLETELY UNDERSTOOD.

Amara [Ama66] has given a complete characterization of the (infinitely many) limit points of the Pisot numbers less than 2. Talmoudi [Tal78] gave the surprising result that for any of these limit points, there is some small neighbourhood around the limit point such that all Pisot numbers in the neighbourhood can be completely determined using highly structured sequences of polynomials.<sup>3</sup>

For example, the smallest limit point of the Pisot numbers is  $\phi = (1 + \sqrt{5})/2$ , the root of  $f(x) = x^2 - x - 1$ . According to Talmoudi's classification, any Pisot number sufficiently close to  $\phi$  must be the root of the a polynomial of the form  $f(x)x^n + g(x)$ , for some  $n \geq 1$  and  $g(x) \in \{\pm 1, \pm x, \pm(x^2 - 1)\}$ . Conversely, any such polynomial will have a Pisot number as a root, for sufficiently large values of  $n$  (although the polynomial may not be irreducible in general). Furthermore, as  $n$  grows arbitrarily large, the Pisot root of this polynomial will approach the limit point  $\phi$ . These special polynomials described by Amara and Talmoudi give rise to what are called "regular Pisot numbers"; any Pisot number not fitting one of these patterns is called "irregular". The irregular Pisot numbers are much less common than the regular Pisot numbers.

## PISOT NUMBERS CAN BE FOUND ALGORITHMICALLY.

Boyd [Boy78, Boy85, Boy84] has presented a remarkable algorithm that deterministically finds all Pisot numbers within any interval  $[a, b]$  of the real line, and it is able to detect and compensate for any limit points of  $S$  that may occur there. Boyd's algorithm, which was developed over the course of three papers, is particularly useful for finding the irregular Pisot numbers, and has marked a big achievement to further the study of Pisot numbers and related areas. Due to this algorithm, along with the other facts known about  $S$ , the set of Pisot numbers is very well understood, and Pisot numbers can be obtained very easily.

<sup>2</sup>Salem's proof in 1944 that  $S$  is closed was a strong motivation to continue the study of Pisot numbers, and Pisot later mentioned that he called the set  $S$  in order to honour Salem for this contribution.

<sup>3</sup>Although Amara and Talmoudi described their highly nontrivial classification of Pisot number sequences in French, the main results have been summarized in various English papers, for example, by Hare [Har07] and Boyd [Boy96].

## 4 OTHER APPLICATIONS OF PISOT NUMBERS

It turns out that Pisot numbers are useful for more than just generating almost integers. In fact, the Pisot numbers have broad applications that arise in a variety of different areas of mathematics, making them a rich class of objects to study. Some examples are as follows.

### SALEM NUMBERS.

Closely related to the set of Pisot numbers is the set  $T$  of Salem numbers.<sup>4</sup> A Salem number is an algebraic integer whose Galois conjugates are all less than or equal to 1 in absolute value, but with at least one root having an absolute value of exactly 1. Although the definition is very similar to that of Pisot numbers, the set of Salem numbers is much less understood than  $S$ . It is known that  $T$  is not closed; a long standing open conjecture is whether or not the set  $T$  has a least element. The currently known smallest element is the root of a degree ten polynomial found by Lehmer [Leh33], but there is no proof that a smaller one does not exist.<sup>5</sup> Salem numbers exhibit a close relation with the Pisot numbers in that every Pisot number is a limit point for a sequence of Salem numbers. However, whether these are the only limit points of  $T$ , like so many other questions concerning Salem numbers, remains unknown [BP90, Boy77].

### MAHLER MEASURE PROBLEMS.

The Mahler measure of a polynomial is the product of all of the complex roots of the polynomial that have absolute value larger than 1. The problem of finding polynomials of very small Mahler measure has interested mathematicians for a long time, and continues to be an active area of study [Mos98]. In particular, the Mahler measure of a minimal polynomial of a Pisot or Salem root  $\alpha$  is always equal to  $\alpha$ , so in this case the problem reduces to finding small Pisot and Salem numbers. The smallest Pisot number is known, as mentioned in Section 3; however, the smallest Salem number (if it exists) is not known, and the study of Mahler measures has therefore motivated the search for the smallest element of  $T$ .

### BETA EXPANSIONS.

Rényi [Rén57] introduced the notion of beta expansions as a number representation system, where numbers are written not using base 10, but base  $\beta$  where  $\beta$  may not necessarily be an integer. It was found that surprising things happen when the base of representation is not a whole number; for example, expansions are frequently not unique, and expansions of rational numbers may neither terminate nor repeat. In general, the expansions are chaotic and unpredictable; however, when the base  $\beta$  is chosen to be a Pisot number, then the expansions are much more well behaved. There has been a lot of study in identifying the patterns that arise in these expansions when the base is chosen to be a Pisot number [Bas02, HT08, Pan11].

### FRACTAL TILINGS, QUASICRYSTALS, AND MORE.

Pisot numbers have many applications in dynamical systems, mainly due to the nonuniform distribution of powers of Pisot numbers modulo one. The patterns in the beta expansions involving Pisot numbers can be used to generate fractal tilings of the plane [AI01]. More recently, Pisot numbers have been used to study the aperiodic tilings of quasicrystals [EF05].

<sup>4</sup>Although it was a nice gesture for Pisot to name his set of numbers  $S$  after Salem, this convention made notation unnecessarily awkward when Salem introduced his own related class of numbers in 1945.

<sup>5</sup>To date, no Salem number smaller than 1.176... , the root of the polynomial  $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ , is known. Despite extensive computer searches, the record still belongs to the polynomial Lehmer found using hand calculations in 1933.

## 5 CONCLUSIONS

The main goal of this paper was to outline a quick and easy method for producing non-integer values that were unusually close to whole numbers. In doing so, we were able to get a glimpse at the structure and properties of the set of Pisot numbers. Although these numbers are useful for generating large quantities of almost integers, we have seen that their study is rich and interesting in its own right, and that the applications of Pisot numbers in mathematics are broad.

There are many unanswered questions related to Pisot numbers, particularly ones involving the related set of Salem numbers. There is also a lot of room for extended study in the applications and properties of these numbers.

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