INTEGERS WITH A PREDETERMINED PRIME FACTORIZATION

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Abstract. A classic question in analytic number theory is to find asymptotics for $\sigma_k(x)$ and $\pi_k(x)$, the number of integers $n \leq x$ with exactly k prime factors, where $\pi_k(x)$ has the added constraint that all the factors are distinct. This problem was originally resolved by Landau in 1900, and much work was subsequently done where k is allowed to vary. In this paper we look at a similar question about integers with a specific prime factorization. Given $\boldsymbol{\alpha} \in \mathbb{N}^k$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ let $\sigma_{\boldsymbol{\alpha}}(x)$ denote the number of integers of the form $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where the p_i are not necessarily distinct, and let $\pi_{\boldsymbol{\alpha}}(x)$ denote the same counting function with the added condition that the factors are distinct. Our main result is asymptotics for both of these functions.

1. Introduction

One of the major problems in the 19th century was to find the growth rate of the number of primes less then x, that is the function

$$\pi(x) := \sum_{p \leq x} 1.$$

In 1797, Legendre conjectured that $\pi(x)$ is asymptotic to $\frac{x}{\log x}$, written as $\pi(x) \sim \frac{x}{\log x}$, which means that we have the limit

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1.$$

Although a more precise conjecture was given by Gauss, little progress was made over the next 50 years. In 1848 and 1850, Chebyshev made several contributions, and managed to prove weaker upper and lower bounds. A major breakthrough occurred in 1859, when Riemann published his seminal paper, "On the Number of Primes Less Than a Given Magnitude," in which he outlined a proof of Legendre's conjecture using complex analysis and the zeta function. In 1896, 99 years after Legendre made his conjecture, Hadamard and de la Vallée Poussin rigorously completed Riemann's outline, proving what is known today as the prime number theorem [3]. In particular, we can write down the explicit error term :

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \tag{1.1}$$

but to be more precise than this we would need to introduce the function from Gauss's conjecture.

A natural follow up question is whether or not we have similar asymptotics for the number of integers with exactly k prime factors. There are two reasonable ways to define the counting function; let $\sigma_k(x)$ denote the number of integers less then x with exactly k prime factors, and let $\pi_k(x)$ be the same but with the added constraint that the k prime factors must be distinct. For convenience, we also define the sets $\mathcal{P}_k^\sigma = \{n : n = p_1 \cdots p_k\}$ and $\mathcal{P}_k^\pi = \{n : n = p_1 \cdots p_k \text{ where } i \neq j \Rightarrow p_i \neq p_j\}$, so that we may write

$$\sigma_k(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{P}_k^\sigma}} 1 \quad \text{and} \quad \pi_k(x) = \sum_{\substack{n \leq x \\ n \in \mathcal{P}_k^\pi}} 1.$$

In 1900 by Landau [2] found the growth rate of these functions, and he proved that for fixed k we have

$$\pi_k(x) \sim \sigma_k(x) \sim \frac{x\,(\log\log x)^{k-1}}{(k-1)!\log x}. \tag{1.2}$$

E. M. Wright then gave a short elementary proof of this in 1954 [4]. Heuristically we might expect this kind of asymptotic since $\sum_{k=1}^{\infty} \sigma_k(x) = \lfloor x \rfloor$, and if we could ignore the error term and sum over all $k \leq \log x$, we would arrive back at this equality again as

$$\sum_{k=1}^{\infty} \sigma_k(x) \approx \sum_{k=1}^{\infty} \frac{x\,(\log\log x)^{k-1}}{(k-1)!\log x} = \frac{x}{\log x}\sum_{k=0}^{\infty} \frac{(\log\log x)^{k}}{k!} = x.$$

Note that even though this works out, the heuristic is not entirely reliable. It seems to suggest that $\sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$ even when k varies with x, which is not true when $k \approx \log\log x$ [1]. In his paper, Landau also gave explicit error terms, and showed that for $k \geq 2$

$$\sigma_k(x) = \frac{x\,(\log\log x)^{k-1}}{(k-1)!\log x} + O\left(\frac{x\,(\log\log x)^{k-2}}{\log x}\right) \tag{1.3}$$

and

$$\pi_k(x) = \frac{x\,(\log\log x)^{k-1}}{(k-1)!\log x} + O\left(\frac{x\,(\log\log x)^{k-2}}{\log x}\right) \tag{1.4}$$

where the notation $O(f(x))$ means that the error term is bounded in absolute value by some constant multiple of $f(x)$. (Although seperated on different lines, note that the above asymptotics are indeed the same.) In this paper we are interested in something very similar, which is counting the number of integers of a particular shape, integers of the form $p_1^{\alpha_1}\cdots p_n^{\alpha_n}$ where the α_i are fixed exponents. For example, we may ask how many integers of the form pq^3 are there less than x. To discuss this problem, we begin by introducing some notation. Given a vector $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_k) \in \mathbb{N}^k$, define $\sigma_{\boldsymbol{\alpha}}(x)$ to be the number of integers $n \le x$ of the form $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, allowing prime repetitions, and $\pi_{\boldsymbol{\alpha}}(x)$ to be the number without prime repetitions. If we set $\mathcal{P}^{\sigma}_{\boldsymbol{\alpha}} = \{n :\ n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}\}$, and $\mathcal{P}^{\pi}_{\boldsymbol{\alpha}} = \{n :\ n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ where $i \ne j \Rightarrow p_i \ne p_j\}$, then as was done for $\pi_k(x)$, and $\sigma_k(x)$, we can rewrite these counting functions as

$$\sigma_{\boldsymbol{\alpha}}(x) = \sum_{\substack{n\le x\\ n\in\mathcal{P}^{\sigma}_{\boldsymbol{\alpha}}}} 1 \quad\text{and}\quad \pi_{\boldsymbol{\alpha}}(x) = \sum_{\substack{n\le x\\ n\in\mathcal{P}^{\pi}_{\boldsymbol{\alpha}}}} 1.$$

Our goal is to provide asymptotics for $\sigma_{\boldsymbol{\alpha}}(x)$ and $\pi_{\boldsymbol{\alpha}}(x)$, and our main theorem is:

Theorem 1. *Let r, α be positive integers. Suppose we have a vector of the form $\boldsymbol{\alpha} = (\alpha,\cdots,\alpha,\alpha_1,\cdots,\alpha_r) \in \mathbb{N}^{k+r}$, where $k > 0$ is the multiplicity of α, and where $\alpha < \alpha_i$ for all i. Then if $\boldsymbol{\beta} = (\alpha_1,\cdots,\alpha_r) \in \mathbb{N}^r$, we have*

$$\sigma_{\boldsymbol{\alpha}}(x) \sim \sigma_k\left(x^{\frac{1}{\alpha}}\right)\sum_{n\in\mathcal{P}^{\sigma}_{\boldsymbol{\beta}}} n^{-\frac{1}{\alpha}}$$

and

$$\pi_{\boldsymbol{\alpha}}(x) \sim \sigma_k\left(x^{\frac{1}{\alpha}}\right)\sum_{n\in\mathcal{P}^{\pi}_{\boldsymbol{\beta}}} n^{-\frac{1}{\alpha}}.$$

The above theorem tells us that the higher powers introduce a constant factor into the asymptotic since both of the series $\sum_{n\in\mathcal{P}^{\sigma}_{\boldsymbol{\beta}}} n^{-\frac{1}{\alpha}}$ and $\sum_{n\in\mathcal{P}^{\pi}_{\boldsymbol{\beta}}} n^{-\frac{1}{\alpha}}$ converge absolutely. The convergence of these series follows from the fact that $\frac{\alpha_i}{\alpha} > 1$ along with equation 2.1 in the next section. In particular, returning to our previous example of counting the number of integers of the form pq^3 less than x, we have that $\mathcal{P}^{\pi}_{\boldsymbol{\beta}} = \mathcal{P}^{\sigma}_{\boldsymbol{\beta}} = \{p^3 :\ p \text{ is prime}\}$, and hence

$$\pi_{(1,3)}(x) \sim \sigma_{(1,3)}(x) \sim \frac{x}{\log x}\sum_p \frac{1}{p^3} = \frac{x}{\log x}P(3)$$

where $P(s)=\sum_p p^{-s}$ is the prime zeta function. We can ask whether the constant can always be rewritten as a product of prime zeta functions, and this is answered by the following theorem:

Theorem 2. *Suppose we are given $\alpha<\alpha_1\leq\cdots\leq\alpha_r$, and that for any choice of $\epsilon_i\in\{-1,0,1\}$, we have $\sum_i\epsilon_i\alpha_i=0$ implies $\epsilon_i=0$ for every i. Then*

$$\sum_{n\in\mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} n^{-\frac{1}{\alpha}}=\prod_{i=1}^{r}P\left(\frac{\alpha_i}{\alpha}\right)$$

where $P(s)=\sum_p p^{-s}$ is the prime zeta function. This is equivalent to the condition that every $n\in\mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$, where $\boldsymbol{\beta}=(\alpha_1,\ldots,\alpha_r)$, has a unique representation as $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$.

For example, the above two theorems imply that the number of integers of the form $n=p_1p_2p_3^3p_4^5p_5^{19}$, with $n\leq x$, will be asymptotic to

$$\sigma_2\left(x\right)P(3)P(5)P(19)\sim\frac{x\log\log x}{\log x}P(3)P(5)P(19).$$

2. The Main Result

It is very important to split up the smallest power, as this is contributes the most to the sum. Throughout this section, we write our vector of exponents as $\boldsymbol{\alpha}=(\alpha,\cdots,\alpha,\alpha_1,\cdots,\alpha_r)\in\mathbb{N}^{k+r}$, with $1\leq\alpha<\alpha_1\leq\cdots\leq\alpha_r$, where $k>0$ is the multiplicity of α, and let $\boldsymbol{\beta}=(\alpha_1,\cdots,\alpha_r)\in\mathbb{N}^r$. To start, we provide a simple upper bound for $\sigma_{\boldsymbol{\beta}}(x)$. Notice that

$$\pi_{\boldsymbol{\beta}}(x)\leq\sigma_{\boldsymbol{\beta}}(x)=\sum_{\substack{m\leq x\\ m\in\mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}}1\leq\sum_{p_1^{\alpha_1}\cdots p_r^{\alpha_r}\leq x}1,$$

where the right hand sum ranges over all vectors of primes of length r satisfying $p_1^{\alpha_1}\cdots p_r^{\alpha_r}\leq x$. Since $\alpha_1\leq\alpha_i$ for all i, and $p_1^{\alpha_1}\cdots p_r^{\alpha_r}\leq x$ implies that $p_1^{\alpha_1}p_2^{\alpha_1}\cdots p_r^{\alpha_1}\leq x$, we see that replacing every exponent by α_1 only increases the sum. Then using 1.2 we have

$$\pi_{\boldsymbol{\beta}}(x)\leq\sigma_{\boldsymbol{\beta}}(x)\leq\sum_{p_1\cdots p_r\leq x^{\frac{1}{\alpha_1}}}1=O\left(x^{\frac{1}{\alpha_1}}\frac{(\log\log x)^{r-1}}{\log x}\right).\tag{2.1}$$

The following subsection is devoted to examining $\sigma_{\boldsymbol{\alpha}}(x)$. The key will be using the hyperbola method, and most of the lemmas will apply identically to the proof for $\pi_{\boldsymbol{\alpha}}(x)$.

2.1. $\sigma_{\boldsymbol{\alpha}}(x)$**.** Each integer $n \in \mathcal{P}_{\boldsymbol{\alpha}}^{\sigma}$ has one part in $\mathcal{P}_k^{\sigma}$, and one part in $\mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$, and our goal will be to split it up between these two to better understand $\sigma_{\boldsymbol{\alpha}}(x)$. With this in mind, we might expect

$$\sigma_{\boldsymbol{\alpha}}(x) \approx \sum_{\substack{mn^{\alpha} \leq x \\ n \in \mathcal{P}_k^{\sigma}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} 1.$$

However, this will not be an exact equality as an integer $k \leq x$ with $k \in \mathcal{P}_{\boldsymbol{\alpha}}^{\sigma}$ may have more than one representation of the form $k = mn^{\alpha}$ with $n \in \mathcal{P}_k^{\sigma}$, $m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$. Since $k \in \mathcal{P}_{\boldsymbol{\alpha}}^{\sigma}$ can have at most one representation of the form $k = mn^{\alpha}$ with $n \in \mathcal{P}_k^{\pi}$, $m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$, we have the inequalities

$$\sum_{\substack{mn^{\alpha} \leq x \\ n \in \mathcal{P}_k^{\pi}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} 1 \leq \sigma_{\boldsymbol{\alpha}}(x) \leq \sum_{\substack{mn^{\alpha} \leq x \\ n \in \mathcal{P}_k^{\sigma}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} 1.$$

Rewriting so that we first sum over m, this is

$$\sum_{\substack{mn^{\alpha} \leq x \\ n \in \mathcal{P}_k^{\sigma}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} 1 = \sum_{\substack{m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \sum_{\substack{n^{\alpha} \leq \frac{x}{m} \\ n \in \mathcal{P}_k^{\sigma}}} 1 = \sum_{\substack{m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \sigma_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right)$$

and we have that

$$\sum_{\substack{m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \pi_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right) \leq \sigma_{\boldsymbol{\alpha}}(x) \leq \sum_{\substack{m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \sigma_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right). \tag{2.2}$$

Our first goal will be to remove all of the terms from the sum with $m \geq \frac{x}{(\log x)^C}$ for some constant $C > 2$, without introducing large error. For example, we could take $C = 3$ to prove the asymptotic. However to achieve the optimal error term we need something of the form $C = 2\alpha\alpha_1 + 1$, a choice which will become clear later on. Note that we need only bound this sum for $\sigma_k(x)$, since $\pi_k(x) \leq \sigma_k(x)$, and this is covered by the following lemma.

Lemma 3. *For $C > 1$ we have that*

$$\sum_{\substack{(\log x)^C < m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \sigma_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right) = O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^{(C-1)\left(1-\frac{\alpha}{\alpha_1}\right)}}\right).$$

Proof. We may change the order of summation and write

$$\sum_{\substack{(\log x)^C \leq m \leq x \\ m \in \mathcal{P}^{\sigma}_{\boldsymbol{\beta}}}} \sigma_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right) = \sum_{\substack{(\log x)^C \leq m \leq x \\ m \in \mathcal{P}^{\sigma}_{\boldsymbol{\beta}}}} \sum_{\substack{n^{\alpha} \leq \frac{x}{m} \\ n \in \mathcal{P}^{\sigma}_{k}}} 1$$

$$= \sum_{\substack{n^{\alpha} \leq \frac{x}{(\log x)^C} \\ n \in \mathcal{P}^{\sigma}_{k}}} \sum_{\substack{(\log x)^C \leq m \leq \frac{x}{n^{\alpha}} \\ m \in \mathcal{P}^{\sigma}_{\boldsymbol{\beta}}}} 1.$$

Using 2.1 this is bounded above by

$$\sum_{\substack{n^{\alpha} \leq \frac{x}{(\log x)^C} \\ n \in \mathcal{P}^{\sigma}_{k}}} \sum_{\substack{m \leq \frac{x}{n^{\alpha}} \\ m \in \mathcal{P}^{\sigma}_{\boldsymbol{\beta}}}} 1 = \sum_{\substack{n^{\alpha} \leq \frac{x}{(\log x)^C} \\ n \in \mathcal{P}^{\sigma}_{k}}} O\left(\frac{x^{\frac{1}{\alpha_1}}}{n^{\frac{\alpha}{\alpha_1}}} \frac{(\log\log(x/n^{\alpha}))^{r-1}}{\log(x/n^{\alpha})}\right)$$

$$= O\left(x^{\frac{1}{\alpha_1}} (\log\log x)^{r-1} \sum_{\substack{n^{\alpha} \leq \frac{x}{(\log x)^C} \\ n \in \mathcal{P}^{\sigma}_{k}}} \frac{1}{n^{\frac{\alpha}{\alpha_1}}}\right).$$

Taking the trivial bound, the inner sum becomes

$$\sum_{\substack{n^{\alpha} \leq \frac{x}{(\log x)^C} \\ n \in \mathcal{P}^{\sigma}_{k}}} \frac{1}{n^{\frac{\alpha}{\alpha_1}}} \leq \sum_{n \leq \frac{x^{\frac{1}{\alpha}}}{\log^{\frac{C}{\alpha}} x}} \frac{1}{n^{\frac{\alpha}{\alpha_1}}} = O\left(\left(\frac{x^{\frac{1}{\alpha}}}{\log^C x}\right)^{-\frac{\alpha}{\alpha_1}-1}\right)$$

$$= O\left(\frac{1}{(\log x)^{C\left(1-\frac{\alpha}{\alpha_1}\right)}}\right),$$

so that we have the upper bound

$$O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^{(C-1)\left(1-\frac{\alpha}{\alpha_1}\right)}} \frac{(\log\log x)^{r-1}}{(\log x)^{1-\frac{\alpha}{\alpha_1}}}\right) = O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^{(C-1)\left(1-\frac{\alpha}{\alpha_1}\right)}}\right).$$

$\square$

Combining 2.2 along with Lemma 3 and Landau's estimates 1.3, 1.4 for $k > 1$ yields

(2.3)
$$\sigma_{\boldsymbol{\alpha}}(x) = \frac{1}{(k-1)!} \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \alpha \frac{x^{\frac{1}{\alpha}} \left(\log\left(\frac{1}{\alpha}\log\left(\frac{x}{m}\right)\right)\right)^{k-1}}{m^{\frac{1}{\alpha}} \log\left(\frac{x}{m}\right)}$$
$$+ O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^{(C-1)\left(1-\frac{\alpha}{\alpha_1}\right)}} + \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \frac{x^{\frac{1}{\alpha}} \left(\log\left(\frac{1}{\alpha}\log\left(\frac{x}{m}\right)\right)\right)^{k-2}}{m^{\frac{1}{\alpha}} \log\left(\frac{x}{m}\right)} \right),$$

and for $k = 1$ by 1.1, the prime number theorem, we have

$$\sigma_{\boldsymbol{\alpha}}(x) = \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \alpha \frac{x^{\frac{1}{\alpha}}}{m^{\frac{1}{\alpha}} \log\left(\frac{x}{m}\right)}$$
(2.4)
$$+ O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^{(C-1)\left(1-\frac{\alpha}{\alpha_1}\right)}} + \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \frac{x^{\frac{1}{\alpha}}}{m^{\frac{1}{\alpha}} \log^2\left(\frac{x}{m}\right)} \right).$$

If we write $\left(\log\left(\frac{1}{\alpha}\log\left(\frac{x}{m}\right)\right)\right)^{k-1} = \left(\log\log\left(\frac{x}{m}\right) - \log\alpha\right)^{k-1}$ and then expand using the binomial theorem, all of the terms will be consumed by the error term except for the one with $\left(\log\log\left(\frac{x}{m}\right)\right)^{k-1}$, which allows us to change the main term in the above to

$$\frac{1}{(k-1)!} \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \alpha \frac{x^{\frac{1}{\alpha}} \left(\log\log\left(\frac{x}{m}\right)\right)^{k-1}}{m^{\frac{1}{\alpha}} \log\left(\frac{x}{m}\right)}. \tag{2.5}$$

We may clean up the error terms by bounding each part of the sum from above. Since $m \leq (\log x)^C$, $\frac{1}{\log\left(\frac{x}{m}\right)}$ is bounded above by

$$\frac{1}{\log\left(\frac{x}{(\log x)^C}\right)} = \frac{1}{\log(x) - C\log\log x} = \frac{1}{\log x} + O\left(\frac{\log\log x}{\log^2 x}\right).$$

We also have the trivial bounds

$$\log\left(\frac{1}{\alpha}\log\left(\frac{x}{m}\right)\right) \leq \left(\log\left(\log\left(x\right)\right)\right),$$

and

$$\sum_{\substack{m \leq (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}}} \frac{1}{m^{\frac{1}{\alpha}}} \leq \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}} m^{-\frac{1}{\alpha}}$$

since the right hand side is a convergent series. Combining these, for integers $A \geq 0$, $B > 1$ we have that

$$\sum_{\substack{m \leq (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}}} \frac{x^{\frac{1}{\alpha}}\left(\log\left(\frac{1}{\alpha}\log\left(\frac{x}{m}\right)\right)\right)^A}{m^{\frac{1}{\alpha}}\log^B\left(\frac{x}{M}\right)} = O\left(\frac{x^{\frac{1}{\alpha}}\left(\log\log x\right)^A}{\log^B(x)}\right), \tag{2.6}$$

which gives an upper bound on the error term in both cases, $k = 1$ and $k > 1$. The following lemma allows us to deal with the main term:

Lemma 4. *For $C > 1$, we have that*

$$\sum_{\substack{m \leq (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}}} \frac{\left(\log\log\left(\frac{x}{m}\right)\right)^{k-1}}{m^{\frac{1}{\alpha}}\log\left(\frac{x}{m}\right)} = \frac{\left(\log\log(x)\right)^{k-1}}{\log x} \sum_{\substack{m \leq (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}}} m^{-\frac{1}{\alpha}} + O\left(\frac{(\log\log x)^{k-1}}{\log^2 x}\right).$$

Proof. First, note that we have the bounds

$$\frac{1}{\log(x)} \leq \frac{1}{\log\left(\frac{x}{m}\right)} \leq \frac{1}{\log\left(\frac{x}{\log x}\right)}$$

and

$$\left(\log\log\left(\frac{x}{\log x}\right)\right)^{k-1} \leq \left(\log\log\left(\frac{x}{m}\right)\right)^{k-1} \leq \left(\log\log(x)\right)^{k-1}.$$

Using power series expansions we may write

$$\frac{1}{\log\left(\frac{x}{\log x}\right)} = \frac{1}{\log(x)\left(1 - \frac{\log\log x}{\log x}\right)} = \frac{1}{\log x} + O\left(\frac{\log\log x}{\log^2 x}\right)$$

and

$$\left(\log\log\left(\frac{x}{\log x}\right)\right)^{k-1} = \left(\log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right)\right)^{k-1} = (\log\log x)^{k-1} + O\left(\frac{(\log\log x)^{k-1}}{\log x}\right).$$

Then 2.6 implies that

$$\sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} \frac{\left(\log\log\left(\frac{x}{m}\right)\right)^{k-1}}{m^{\frac{1}{\alpha}} \log\left(\frac{x}{m}\right)} = \frac{(\log\log x)^{k-1}}{\log x} \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}} + O\left(\frac{(\log\log x)^{k-1}}{\log^2 x}\right).$$

□

Let $C = 2\alpha\alpha_1 + 1$ so that $(C-1)\left(1 - \frac{\alpha}{\alpha_1}\right) = 2\alpha(\alpha_1 - \alpha) \ge 2$. Upon combining 2.3, 2.5, 2.6, and lemma 4 for $k > 1$ we obtain

$$\sigma_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}} (\log\log x)^{k-1}}{(k-1)! \log x} \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}} + O\left(x^{\frac{1}{\alpha}} \frac{(\log\log x)^{k-2}}{\log x}\right). \tag{2.7}$$

Similarly, 2.4, 2.6, and lemma 4 together yield

$$\sigma_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}}}{\log x} \sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}} + O\left(\frac{x^{\frac{1}{\alpha}}}{\log^2 x}\right)$$

for $k = 1$. To deal with the last sum, write

$$\sum_{\substack{m \le (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}} = \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} m^{-\frac{1}{\alpha}} - \sum_{\substack{m > (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}}.$$

Applying summation by parts, we have that

$$\begin{aligned}\sum_{\substack{m > (\log x)^C \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}}} m^{-\frac{1}{\alpha}} &= \int_{(\log x)^C}^{\infty} t^{-\frac{1}{\alpha}} d\left(\sigma_{\boldsymbol{\beta}}(t)\right) \\ &= t^{-\frac{1}{\alpha}} \sigma_{\boldsymbol{\beta}}(t)\Big|_{(\log x)^C}^{\infty} + \frac{1}{\alpha} \int_{(\log x)^C}^{\infty} t^{-\frac{1}{\alpha}-1} \sigma_{\boldsymbol{\beta}}(t) dt.\end{aligned}$$

Then by 2.1 this becomes

$$O\left((\log x)^{C\left(\frac{1}{\alpha_1}-\frac{1}{\alpha}\right)} (\log\log\log x)^{r-1}\right) = O\left(\frac{1}{(\log x)^2}\right)$$

since $C\left(\frac{1}{\alpha_1}-\frac{1}{\alpha}\right) = -2 + \left(\frac{1}{\alpha_1}-\frac{1}{\alpha}\right)$. Thus for $k > 1$ we have

$$\sigma_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}} (\log\log x)^{k-1}}{(k-1)! \log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} m^{-\frac{1}{\alpha}} + O\left(x^{\frac{1}{\alpha}} \frac{(\log\log x)^{k-2}}{\log x}\right), \tag{2.8}$$

and for $k = 1$,

$$\sigma_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}}}{\log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} m^{-\frac{1}{\alpha}} + O\left(\frac{x^{\frac{1}{\alpha}}}{(\log x)^2}\right). \tag{2.9}$$

This yields the desired asymptotic

$$\sigma_{\boldsymbol{\alpha}}(x) \sim \alpha \frac{x^{\frac{1}{\alpha}} (\log\log x)^{k-1}}{(k-1)! \log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} m^{-\frac{1}{\alpha}}, \tag{2.10}$$

and since

$$\sigma_k\left(x^{\frac{1}{\alpha}}\right) \sim \alpha \frac{x^{\frac{1}{\alpha}} (\log\log x)^{k-1}}{(k-1)! \log x}$$

by Landau's estimates 1.2, we conclude that

$$\sigma_{\boldsymbol{\alpha}}(x) \sim \sigma_k\left(x^{\frac{1}{\alpha}}\right) \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} m^{-\frac{1}{\alpha}}, \tag{2.11}$$

proving the first part of Theorem 1.

2.2. $\pi_{\boldsymbol{\alpha}}(x)$**.** To prove the same result for $\pi_{\boldsymbol{\alpha}}(x)$, we start again by splitting integers $n \in \mathcal{P}_{\boldsymbol{\alpha}}^{\sigma}$ into two parts, one in $\mathcal{P}_k^{\pi}$, and one in $\mathcal{P}_{\boldsymbol{\beta}}^{\pi}$. With this in mind we consider

$$\sum_{\substack{n^\alpha m \le x \\ n \in \mathcal{P}_k^{\pi}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}}} 1.$$

This will be strictly larger then $\pi_{\boldsymbol{\alpha}}(x)$ since n and m may have prime factors in common. (Note that since all factors are distinct, we cannot have multiple representations $k = mn$.) However, we can throw out all of the terms for which $\gcd(m, n) > 1$ without affecting the asymptotic. Write $n = q_1 \cdots q_k$, and $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. If $\gcd(m, n) > 1$, then we must have $q_i = p_j$ for some i, j. The set of all tuples with $q_i = p_j$ is bounded above by

$$\sigma_{\boldsymbol{\alpha}_{i,j}}(x)$$

where $\boldsymbol{\alpha}_{i,j} = (\alpha, \ldots, \alpha, \alpha_1, \cdots, (\alpha_j + \alpha), \cdots, \alpha_r) \in \mathbb{N}^{k-1+r}$ and we have $k-1$ copies of α. In particular, by 2.10, we see that

$$\sigma_{\boldsymbol{\alpha}_{i,j}}(x) = O\left(x^{\frac{1}{\alpha}} \frac{(\log\log x)^{k-2}}{\log x}\right)$$

for $k > 1$, and

$$\sigma_{\boldsymbol{\alpha}_{i,j}}(x) = O_\epsilon\left(x^{\frac{1}{\alpha_1}+\epsilon}\right)$$

for any $\epsilon > 0$ when $k = 1$. Since there are at most $k \cdot r$ possible pairs (i, j), it follows that for $k > 1$

$$\pi_{\boldsymbol{\alpha}}(x) = \sum_{\substack{n^\alpha m \le x \\ n \in \mathcal{P}_k^{\pi}, m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}}} 1 + O\left(x^{\frac{1}{\alpha}} \frac{(\log\log x)^{k-2}}{\log x}\right),$$

and a similar error term as before when $k = 1$. The main term may be rewritten as

$$\sum_{\substack{m \le x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}}} \sum_{\substack{n^\alpha \le \frac{x}{m} \\ n \in \mathcal{P}_k^{\pi}}} 1 = \sum_{\substack{m \le x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}}} \pi_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right),$$

and from here, following through the exact same sequence of steps and lemmas from the previous section will yield

$$\sum_{\substack{m \leq x \\ m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}}} \pi_k\left(\left(\frac{x}{m}\right)^{\frac{1}{\alpha}}\right) \sim \alpha \frac{x^{\frac{1}{\alpha}}(\log\log x)^{k-1}}{(k-1)!\log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}} m^{-\frac{1}{\alpha}}.$$

All of the upper bounds for $\sigma_{\boldsymbol{\alpha}}(x)$ still apply to $\pi_{\boldsymbol{\alpha}}(x)$, and the only change is that we are summing over $\mathcal{P}_{\boldsymbol{\beta}}^{\pi}$ rather then $\mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$, which is why the final sum is different. Using 1.2, we get that

$$\pi_{\boldsymbol{\alpha}}(x) \sim \pi_k\left(x^{\frac{1}{\alpha}}\right) \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}} m^{-\frac{1}{\alpha}}, \tag{2.12}$$

proving the second part of Theorem 1. If the error term is kept throughout the above computations, we get the more precise

$$\pi_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}}(\log\log x)^{k-1}}{(k-1)!\log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}} m^{-\frac{1}{\alpha}} + O\left(x^{\frac{1}{\alpha}} \frac{(\log\log x)^{k-2}}{\log x}\right) \tag{2.13}$$

when $k > 1$, and

$$\pi_{\boldsymbol{\alpha}}(x) = \alpha \frac{x^{\frac{1}{\alpha}}(\log\log x)^{k-1}}{(k-1)!\log x} \sum_{m \in \mathcal{P}_{\boldsymbol{\beta}}^{\pi}} m^{-\frac{1}{\alpha}} + O\left(\frac{x^{\frac{1}{\alpha}}}{\log^2 x}\right), \tag{2.14}$$

for $k = 1$.

3. The Constant Factor

Let $\alpha > 0$ be given, let $A = \{\alpha_1, \cdots, \alpha_r\}$ where $\alpha < \alpha_i \leq \alpha_j$ for all i, j, and set set $\boldsymbol{\beta} = (\alpha_1, \ldots, \alpha_r)$. If every $n \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}$ has one and only one representation of the form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then we may decompose the sum as

$$\sum_{n \in \mathcal{P}_{\boldsymbol{\beta}}^{\sigma}} n^{-\frac{1}{\alpha}} = \sum_{p_1}\sum_{p_2} \cdots \sum_{p_r} \left(p_1^{\alpha_1} \cdots p_r^{\alpha_r}\right)^{-\frac{1}{\alpha}}.$$

This equals

$$\left(\sum_{p_1} p_1^{-\frac{\alpha_1}{\alpha}}\right) \cdots \left(\sum_{p_r} p_r^{-\frac{\alpha_r}{\alpha}}\right)$$

which by definition of the prime zeta function, $P(s) = \sum_p p^{-s}$, is

$$\prod_{i=1}^{r} P\left(\frac{\alpha_i}{\alpha}\right).$$

We now show that each integer can be uniquely represented if and only if $\sum_i \epsilon_i \alpha_i = 0$ with $\epsilon_i \in \{-1, 0, 1\}$ implies that every $\epsilon_i = 0$. Suppose we are given ϵ_i, not all zero, with $\sum_i \epsilon_i \alpha_i = 0$. Then we have then we have $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} = \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_l} = M$ for some M where each all of the i_n and j_m are distinct. Setting $p_{i_1} = \cdots = p_{i_k} = p$, and $p_{j_1} = \cdots = p_{j_l} = q$, we will have a factor of $q^M p^M$, and this allows us to permute q and p giving two representations of the same integer. Conversely, if we have two representations of the same integer, then it must be because of a factor of the form $q^M p^M$, which implies that we must have $\sum_i \epsilon_i \alpha_i = 0$ for some non zero choices ϵ_i. This then completes the proof of Theorem 2.

References

[1] Adolf Hildebrand and Gérald Tenenbaum. On the number of prime factors of an integer. *Duke Math. J.*, 56(3):471–501, 1988.

[2] E. Landau. Sur quelques problèmes relatifs à la distribution des nombres premiers. *Bull. Soc. Math. France*, 28:25–38, 1900.

[3] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative number theory. I. Classical theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[4] E. M. Wright. A simple proof of a theorem of Landau. *Proc. Edinburgh Math. Soc. (2)*, 9:87–90, 1954.